THE DECOMPOSITION INTO CELLS OF THE AFFINE

WEYL GROUPS OF TYPE A

J-Y SHI

Thesis submitted for the degree of Doctor of

Philosophy at Warwick University.

April, 1984

Mathematics Institute
University of Warwick
COVENTRY CV4 7AL

ACKNOWLEDGEMENT

First and foremost, I wish to thank my supervisor, Professor R.W. Carter, who has been a constant source of encouragement and ideas, and who has helped me over my difficulties. Further, I am indebted to Professor G. Lusztig, who permitted me to use his recent unpublished results on the cells of the affine Weyl groups, by which the results of my thesis are made more satisfactory. I am also grateful to Ms Terri Moss who has so patiently and cheerfully done the difficult job of typing.

I wish also to acknowledge the financial support of the Chinese government and the Committee of Vice-Chancellors and Principals of the universities of the United Kingdom.

Finally, I would like to thank my family for always having been so encouraging and understanding over the past three years.

ABSTRACT

In [1], Kazhdan and Lusztig introduce the concept of a W-graph for a Coxeter group W. In particular, they define left, right and two-sided cells. These W-graphs play an important role in the representation theory. However, the algorithm given by Kazhdan and Lusztig to compute these cells is enormously complicated. These cells have been worked out only in a very few cases. In the present thesis, we shall find all the left, right and two-sided cells in the affine Weyl group $\tilde{A}_n$ of type A_{n-1}, $n > 2$. Our main results show that each left (resp. right) cell of $\tilde{A}_n$ determines a partition, say λ of n and, is characterized by a λ-tabloid and also by its generalized right (resp. left) τ-invariant. There exists a one-to-one correspondence between the set of two-sided cells of $\tilde{A}_n$ and the set Λ_n of partitions of n. The number of left (resp. right) cells corresponding to a given partition $\lambda \in \Lambda_n$ is equal to $\dfrac{n!}{\prod_{j=1}^{m} \mu_j!}$, where $\mu = \{\mu_1 > \ldots > \mu_m\}$ is the dual partition of λ. Each two-sided cell in $\tilde{A}_n$ is also an RL-equivalence class of $\tilde{A}_n$ and is a connected set. Each left (resp. right) cell in $\tilde{A}_n$ is a maximal left (resp. right) connected component in the two-sided cell of $\tilde{A}_n$ containing it. Let $\mathcal{P}$ be any proper standard parabolic subgroup of $\tilde{A}_n$ isomorphic to the symmetric group S_n. Then the intersection of $\mathcal{P}$ with each two-sided cell of $\tilde{A}_n$ is non-empty and is just a two-sided cell of $\mathcal{P}$. The intersection of $\mathcal{P}$ with each left (resp. right) cell of $\tilde{A}_n$ is either empty or a left (resp. right) cell of $\tilde{A}_n$.

Most of these results were conjectured by Lusztig [2], [3].

CONTENTS

CHAPTER 1 : INTRODUCTION

We shall first define cells of any given Coxeter group and state some of their elementary properties, most of which appear in [1], and then introduce our work.

1.1 CELLS AND SOME OF THEIR ELEMENTARY PROPERTIES

Let $W = \langle W,S\rangle$ be a Coxeter group with S the set of Coxeter generators. There is an associative algebra $\tilde{H}$ over the polynomial ring $\mathbb{Z}[q]$ with basis elements $\{T_w | w \in W\}$ satisfying the following relations:

$$T_w T_{w'} = T_{ww'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w')$$

$$(T_s+1)(T_s-q) = 0, \text{if } s \in S,$$

here $\ell(w)$ is the length of w. We define the Hecke algebra H to be $\tilde{H} \otimes_{\mathbb{Z}[q]} A$, where $A = \mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$.

To construct a representation of H endowed with a special basis, we define a W-graph to be a set of vertices X, with a set Y of edges (an edge is a subset of X consisting of two elements) together with two additional data: for each vertex $x \in X$, we are given a subset I_x of S and, for each ordered pair of vertices y, x such that $\{y,x\} \in Y$, we are given an integer $\mu(y,x) \neq 0$. These data are subject to the following requirements: let E be the free A-module with basis X. Then for any $s \in S$,

$$\tau_s(x) = \begin{cases} -x, & \text{if } s \in I_x \\ qx + q^{\frac{1}{2}} \sum\limits_{\substack{y\in X \\ s\in I_y \\ \{y,x\}\in Y}} \mu(y,x)y, & \text{if } s \notin I_x \end{cases}$$

defines an endomorphism of E (i.e. the sum over y is assumed to be always finite) and there is a unique representation $\phi : H \to \mathrm{End}(E)$ such that $\phi(T_s) = \tau_s$ for each $s \in S$.

We shall construct, for any W, such a graph. First, we give some definitions. Let $a \to \bar{a}$ be the involution of the ring A defined by $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$. This extends to an involution $h \to \bar{h}$ of the ring H, defined by

$$\overline{\Sigma a_w T_w} = \Sigma \bar{a}_w T_{w^{-1}}^{-1}$$

(Note that T_w is an invertible element H, for example, if $s \in S$, we have $T_s^{-1} = q^{-1}T_s + (q^{-1}-1)$). For any $w \in W$, we define $q_w = q^{\ell(w)}$, $\varepsilon_w = (-1)^{\ell(w)}$. Let $<$ be the Bruhat order relation on W [defined in [9]]: We can now state

<u>Theorem 1.1.1</u> <u>For any $w \in W$, there is a unique element $C_w \in H$</u> such that

$$\underline{\bar{C}_w = C_w}$$

$$\underline{C_w = \sum_{y \le w} \varepsilon_y \varepsilon_w q_w^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}}\, T_y}$$

<u>where $P_{y,w}$ is a polynomial in q of degree $\le \frac{1}{2}(\ell(w)-\ell(y)-1)$ for $y < w$ and $P_{w,w} = 1$</u>.

It is well known that for each $y < w$ with $\ell(w)-\ell(y) \le 2$, we have $P_{y,w} = 1$. In the case that W is either an ordinal Weyl group or an affine Weyl group, Lusztig [1] [4] has proved that all coefficients of the polynomial $P_{y,w}$ are non-negative integers.

defines an endomorphism of E (i.e. the sum over y is assumed to be always finite) and there is a unique representation $\phi: \mathcal{H} \to \mathrm{End}(E)$ such that $\phi(T_s) = \tau_s$ for each $s \in S$.

We shall construct, for any W, such a graph. First, we give some definitions. Let $a \to \bar{a}$ be the involution of the ring A defined by $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$. This extends to an involution $h \to \bar{h}$ of the ring $\mathcal{H}$, defined by

$$\overline{\Sigma a_w T_w} = \Sigma \bar{a}_w T_{w^{-1}}^{-1}$$

(Note that T_w is an invertible element $\mathcal{H}$, for example, if $s \in S$, we have $T_s^{-1} = q^{-1}T_s + (q^{-1}-1))$. For any $w \in W$, we define $q_w = q^{\ell(w)}$, $\epsilon_w = (-1)^{\ell(w)}$. Let $<$ be the Bruhat order relation on W [defined in [9]]: We can now state

<u>Theorem 1.1.1</u> <u>For any $w \in W$, there is a unique element $C_w \in \mathcal{H}$</u> such that

$$\underline{\bar{C}_w = C_w}$$

$$\underline{C_w = \sum_{y \leq w} \epsilon_y \epsilon_w q_w^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}}\, T_y}$$

<u>where</u> $P_{y,w}$ <u>is a polynomial in q of degree $\leq \frac{1}{2}(\ell(w)-\ell(y)-1)$ for $y < w$ and</u> $P_{w,w}$ <u>$= 1$</u>.

It is well known that for each $y < w$ with $\ell(w)-\ell(y) < 2$, we have $P_{y,w} = 1$. In the case that W is either an ordinal Weyl group or an affine Weyl group, Lusztig [1] [4] has proved that all coefficients of the polynomial $P_{y,w}$ are non-negative integers.

<u>Definition 1.1.2</u> Given $y, w \in W$, we say that $y \prec w$ if the following conditions are satisfied: $y < w$, $\varepsilon_w = -\varepsilon_y$ and $P_{y,w}$ (given by Theorem 1.1.1) is a polynomial in q of degree exactly $\frac{1}{2}(\ell(w)-\ell(y)-1)$. In this case, the leading coefficient of $P_{y,w}$ is denoted by $\mu(y,w)$. It is a non-zero integer. If $w \prec y$, we set $\mu(w,y) = \mu(y,w)$.

By this definition, it is easily seen that for any $y < w$ with $\ell(w) = \ell(y) + 1$, we have $y \prec w$.

Let W° be the group opposed to W. Then $(W \times W^{\circ}, S \coprod S^{\circ})$ is a Coxeter group. Let Γ_W be the graph with its vertex set $\{w | w \in W\}$ and its edge set $\{\{y,w\} | y \prec w\}$. For each $w \in W$, let $I_w = \mathcal{L}(w) \coprod \mathcal{R}(w)^{\circ} \subset S \coprod S^{\circ}$, where $\mathcal{L}(w) = \{s \in S | sw < w\}$, $\mathcal{R}(w) = \{s \in S | ws < w\}$.

<u>Theorem 1.1.3</u> Γ_W, <u>together with the assignment</u> $w \to I_w$ <u>and with the function</u> μ <u>defined above, is a</u> $W \times W^{\circ}$<u>-graph</u>.

Now given any W-graph Γ, and a subset $S' \subset S$, we can regard Γ as W'-graph (where W' is the subgroup of W generated by S') by replacing the set $I_x \subset S$ for each vertex x of Γ, by the set $I_x \cap S'$. In particular, Γ_W can be regarded as a W-graph and as a W°-graph.

Given any W-graph Γ, we define a preorder relation $\underset{\Gamma}{\leq}$ on the set of vertices Γ as follows: we say that the vertices x, x' satisfy $x \underset{\Gamma}{\leq} x'$, if there exists a sequence of vertices $x = x_0, x_1, \ldots, x_t = x'$ such that for each i $(1 \leq i \leq t)$, $\{x_{i-1}, x_i\}$ is an edge of Γ and $I_{x_{i-1}} \not\subset I_{x_i}$. The equivalence relation on the set of vertices corresponding to this preorder is denoted $\underset{\Gamma}{\sim}$ (Thus $x \underset{\Gamma}{\sim} x'$ means that $x \underset{\Gamma}{\leq} x' \underset{\Gamma}{\leq} x$). Each equivalence class, regarded as a full subgraph of Γ (with the same sets I_x and the same function μ) is itself a W-graph. The set of equivalence classes is an ordered set with respect to $\underset{\Gamma}{\leq}$. In the case of the $W \times W^o$-graph Γ_W, the equivalence classes for $\underset{\Gamma}{\sim}$ are called the 2-sided cells of W. When Γ_W is regarded as a W-graph, we shall use the notation $\underset{L}{\leq}$, $\underset{L}{\sim}$ instead of $\underset{\Gamma}{\leq}$, $\underset{\Gamma}{\sim}$; the corresponding equivalence classes are called the left cells of W. When Γ_W is regarded as a W^o-graph, we shall use the notation $\underset{R}{\leq}$, $\underset{R}{\sim}$, instead of $\underset{\Gamma}{\leq}$, $\underset{\Gamma}{\sim}$; the corresponding equivalence classes are called the right cells of W. A minimal non-empty set which is both a union of left cells and a union of right cells is called a RL-equivalence class, written $w \underset{RL}{\sim} y$ if w,y lie in the same RL-equivalence class. Clearly, any 2-sided cell is a union of some RL-equivalence classes.

We now state a property of the preorders $\underset{L}{\leq}$ and $\underset{R}{\leq}$ on W.

<u>Theorem A</u> [1, Proposition 2.4]

(i) <u>If $x \underset{L}{\leq} y$, then $\mathcal{R}(x) \supset \mathcal{R}(y)$. Hence, if $x \underset{L}{\sim} y$, then $\mathcal{R}(x) = \mathcal{R}(y)$.</u>

(ii) <u>If $x \underset{R}{\leq} y$, then $\mathcal{L}(x) \supset \mathcal{L}(y)$. Hence, if $x \underset{R}{\sim} y$, then $\mathcal{L}(x) = \mathcal{L}(y)$.</u> □

Let us fix two generators s,t in S such that st has order 3. Let

$$\mathcal{D}_L(s,t) = \{w \in W \mid \mathcal{L}(w) \cap \{s,t\} \text{ has exactly one element}\}$$

$$\mathcal{D}_R(s,t) = \{w \in W \mid \mathcal{R}(w) \cap \{s,t\} \text{ has exactly one element}\}$$

If $w \in \mathcal{D}_L(s,t)$, then exactly one of the elements sw, tw is in $\mathcal{D}_L(s,t)$, we denote it $*w$. The map $w \to *w$ is an involution of $\mathcal{D}_L(s,t)$. Similarly, we have an involution $w \to w^*$ of $\mathcal{D}_R(s,t)$: w^* is the unique element of $\mathcal{D}_R(s,t) \cap \{ws, wt\}$. Let $\langle s,t \rangle$ be the group of order 6 generated by s,t. We have

Theorem B [1, Theorem 4.2]

Let y,w be two elements in $\mathcal{D}_L(s,t)$.

(i) If $yw^{-1} \notin \langle s,t \rangle$, then we have $y \prec w$ if and only if $*y \prec *w$, and then $\mu(y,w) = \mu(*y,*w)$.

(ii) If $yw^{-1} \in \langle s,t \rangle$, then we have $y \prec w$ if and only if $*w \prec *y$, and then $\mu(y,w) = \mu(*w,*y) = 1$.

Let y,w be two elements in $\mathcal{D}_R(s,t)$.

(iii) If $y^{-1}w \notin \langle s,t \rangle$, then we have $y \prec w$ if and only if $y^* \prec w^*$, and then $\mu(y,w) = \mu(y^*,w^*)$.

(iv) If $y^{-1}w \in \langle s,t \rangle$, then we have $y \prec w$ if and only if $w^* \prec y^*$, and then $\mu(y,w) = \mu(w^*,y^*) = 1$. □

Theorem C [1, Corollary 4.3]

(i) Let y, w be two elements in $\mathcal{D}_L(s,t)$. If $y \underset{R}{\sim} w$, then $*y \underset{R}{\sim} *w$.

(ii) Let y,w be two elements in $\mathcal{D}_R(s,t)$. If $y \underset{L}{\sim} w$, then $y^* \underset{L}{\sim} w^*$. □

We now define an equivalence relation P_L on W generated by $w \underset{P_L}{\sim} *w$ in $\mathcal{D}_L(s,t)$ for some $s,t \in S$ with st having order 3. We call these equivalence classes the P_L-equivalence classes. Similarly, we can define an equivalence relation P_R on W by replacing "$w \underset{P_L}{\sim} *w$ in $\mathcal{D}_L(s,t)$" by "$w \underset{P_R}{\sim} w^*$ in $\mathcal{D}_R(s,t)$", and also define a P-equivalence class on W to be a minimal non-empty set in W which is both a union of P_L-equivalence classes and a union of P_R-equivalence classes.

<u>Theorem D</u>. <u>For any $y, w \in W$.</u>

(i) <u>If $w \underset{P_L}{\sim} y$, then $w \underset{L}{\sim} y$.</u>

(ii) <u>If $w \underset{P_R}{\sim} y$, then $w \underset{R}{\sim} y$.</u>

(iii) <u>If $w \underset{P}{\sim} y$, then $w \underset{RL}{\sim} y$.</u>

<u>Proof:</u> (i) can be reduced to the case when $y = {}^*w$ in $\mathcal{D}_L(s,t)$ for some $s,t \in S$ with st having order 3. Then either $y \prec w$ or $w \prec y$. Since the sets $\mathcal{L}(y) \cap \{s,t\}$ and $\mathcal{L}(w) \cap \{s,t\}$ both contain exactly one element and these elements are distinct, this implies $\mathcal{L}(y) \not\subseteq \mathcal{L}(w)$ and $\mathcal{L}(w) \not\subseteq \mathcal{L}(y)$. So $y \underset{L}{\sim} w$. Similarly for (ii), and (iii) follows from (i), (ii). □

Now we define the generalized right (resp. left) τ-invariant of $w \in W$. [8]

<u>Definition 1.1.4</u> We say $w, y \in W$ are equivalent to order zero, written $w \overset{R}{\underset{o}{\approx}} y$, if $\mathcal{R}(w) = \mathcal{R}(y)$. Inductively, we define equivalence to order n for $n > 1$. We say w,y are equivalent to order n, written $w \overset{R}{\underset{n}{\approx}} y$, if $w \overset{R}{\underset{n-1}{\approx}} y$ and for every $s,t \in S$ with st having order 3 and $y,w \in \mathcal{D}_R(s,t)$, we have $y' \overset{R}{\underset{n-1}{\approx}} w'$, where $y' = y^*$ and $w' = w^*$ in $\mathcal{D}_R(s,t)$. We say w,y have the same generalized right τ-invariant if $w \overset{R}{\underset{n}{\approx}} y$ for any $n > 0$. Similarly, we can define the generalized left τ-invariant of $w \in W$ by replacing R, $\mathcal{R}$ by L, $\mathcal{L}$ in the above definition.

§1.2 A STATEMENT OF OUR MAIN RESULTS

From the above statement, we see that the cells play an important role in the representation theory. However, the algorithm given by Kazhdan and Lusztig to compute these cells is enormously complicated when the order of W gets larger. They have been worked out only in a very few cases. For example we know [5] that in the symmetric groups S_n, $n > 1$, each left (resp. right) cell determines a partition of n. There exists a one-to-one correspondence between the set of two-sided cells of S_n and the set Λ_n of partitions of n. In the present thesis, we shall extend these results to the case of the affine Weyl group A_n of type A_{n-1}, $n > 2$.

The affine Weyl group A_n can be described in several different ways: It can be described as a Coxeter group, as a subgroup of the permutation group on $\mathbb{Z}$ or a group of affine matrices of period n (The precise descriptions of all these will be given in Chapter 2) etc. The last two descriptions lead us to define a map $\sigma: A_n \to \Lambda_n$ by Curtis Greene's procedure [7].

Our main results show that <u>for any $\lambda = \{\lambda_1 > \ldots > \lambda_r\} \in \Lambda_n$ the fibre $\sigma^{-1}(\lambda)$ is a two-sided cell and also an RL-equivalence class of A_n. $\sigma^{-1}(\lambda)$ consists of $\dfrac{n!}{\prod_{j=1}^{m} \mu_j!}$ left (resp. right) cells of A_n, where $\mu = \{\mu_1 > \ldots > \mu_m\}$ is the dual partition of λ. $\sigma^{-1}(\lambda)$ is also a connected set</u> (for the definitions of a connected set as well as a maximal left connected component below, see §15.2). <u>Each left cell in $\sigma^{-1}(\lambda)$ is a maximal left connected component of $\sigma^{-1}(\lambda)$ which can be characterised by some λ-tabloid, where a</u>

λ-tabloid is, by definition, an array of n numbers $\{1,2,\ldots,n\}$ into λ_1 columns such that its t-th column contains μ_t numbers for any t, $1 < t < \lambda_1$. Each left cell is also characterized by its generalized right τ-invariant. Let $\mathcal{P}$ be any proper standard parabolic subgroup of $\tilde{A}_n$ isomorphic to the symmetric group S_n. Then the intersection of $\mathcal{P}$ with any two-sided cell of $\tilde{A}_n$ is non-empty and is just a two-sided cell of $\mathcal{P}$. The intersection of $\mathcal{P}$ with any left (resp. right) cell of $\tilde{A}_n$ is either empty or a left (resp. right) cell of $\mathcal{P}$.

In the present thesis, we mainly regard $\tilde{A}_n$ as a group of affine matrices of period n.

In Chapter 2, we give three different descriptions of the affine Weyl group $\tilde{A}_n$ and some elementary properties for this group. We also define a star operation on an element of $\tilde{A}_n$ which is one of the most important operations in our thesis.

We define a map $\sigma : \tilde{A}_n \to \Lambda_n$ in Chapter 3 by Curtis Green's procedure [7] and show that the fibres of σ are invariant under star operations.

In Chapter 4, we determine some special cells of $\tilde{A}_n$, $n > 2$. In particular, we get all cells of $\tilde{A}_2$. So in the subsequent chapters, we shall always assume $n > 2$.

We define, in Chapter 5, iterated star operations on an element w of $\tilde{A}_n$ and interchanging operations on blocks of w. They are all successions of left star operations. These operations are very important in the proof of our main results.

Then the proof of our main results starts from Chapter 6 and is divided into four main steps.

First we show in Chapters 6 and 7 that for any $\lambda \in \Lambda_n$, $\sigma^{-1}(\lambda)$ is a union of RL-equivalence classes of A_n and for any $w \in \sigma^{-1}(\lambda)$, there exists an element y of N_λ with $y \underset{L}{\tilde{p}} w$, where N_λ is the set of left normalized elements (often simply called normalized elements) of $\sigma^{-1}(\lambda)$.

The second step is the most difficult part in the whole of our proof and includes Chapters 8-11. In this step, we define a map $T: N_\lambda \to C_\lambda$ for some fixed $\lambda \in \Lambda_n$ and then show that for $y, w \in N_\lambda$, $y \underset{L}{\sim} w$ if and only if $T(y) = T(w)$. The proof of this result is achieved bv considering a new kind of operation on an element w of N_λ called a raising operation on a layer of w, which is not in general a succession of star operations but gives an element in the same left cell as w. Since $|C_\lambda| = \dfrac{n!}{\prod_{j=1}^{m} \mu_j!}$ with $\mu = \{\mu_1 > \dots > \mu_m\}$ the dual partition of λ, this confirms Lusztig's first conjecture [2] which says that $\sigma^{-1}(\lambda)$ consists of $\dfrac{n!}{\prod_{j=1}^{m} \mu_j!}$ left (resp. right) cells of A_n for any $\lambda \in \Lambda_n$.

Thirdly, we show in Chapter 12 that $\sigma^{-1}(\lambda)$ is just a single RL-equivalence class of A_n for any $\lambda \in \Lambda_n$.

Finally, by applying a recent result of Lusztig [6], which comes from the deep theory of intersection cohomology, to the affine Weyl group A_n, we verify Lusztig's second conjecture [2] in Chapter 14 which says that $\sigma^{-1}(\lambda)$ is a two-sided cell of A_n for any $\lambda \in \Lambda_n$. This recent result of Lusztig is as follows.

Theorem E. Let W_a be any affine Weyl group. Let z, z' be two elements of W_a which satisfy the following conditions:

(i) Either $z' > z$ or $z' < z$.

(ii) $\mathcal{R}(z') \not\subset \mathcal{R}(z)$ and $\mathcal{L}(z') \not\subset \mathcal{L}(z)$.

Then z and z' are not in the same two-sided cell of W_a. □

On the other hand, in Chapter 13, we give another characterization of any left cell of A_n in terms of generalized right τ-invariant and conclude that any two elements y, w of A_n lie in the same left cell if and only if they have the same generalized right τ-invariant. Let P be any standard parabolic subgroup of A_n isomorphic to the symmetric group S_n. We show in Chapter 13 and 14 that the intersection of P with $\sigma^{-1}(\lambda)$ for any $\lambda \in \Lambda_n$ is non-empty and is just a two-sided cell of P and then verifies Lusztig's third conjecture in the case $W_a = A_n$. This conjecture [3] says that each two-sided cell in W_a meets some proper standard parabolic subgroup of W_a, where W_a is any affine Weyl group. We also show that the intersection of P with any left (resp. right) cell of A_n is either empty or a left (resp. right) cell of P.

In the last chapter, we show that left star operations commute with right star operations on an element of A_n. We also show that each two-sided cell or P-equivalence class in A_n is a connected set and that each left (resp. right) cell or P_L-(resp. P_R-)equivalence class in A_n is a left (resp. right) connected set. Moreover, each left (resp. right) cell in A_n is a maximal left (resp. right) connected component in the two-sided cell containing it.

CHAPTER 2 : ELEMENTARY PROPERTIES OF THE AFFINE WEYL GROUP $\mathcal{A}_n$ OF TYPE A_{n-1}, $n > 2$

In Chapter 1, we have given the definitions of left, right and 2-sided cells as well as RL-equivalence class for an arbitrary Coxeter group. From now on, we shall restrict our attention only to the affine Weyl group $\mathcal{A}_n$ of type A_{n-1}, $n > 2$, and consider how to decompose this group into these equivalence classes.

In this chapter, we shall first give three equivalent descriptions of $\mathcal{A}_n$ and then show some basic properties of the length function $\ell(w)$ and the sets $\mathcal{L}(w)$, $\mathcal{R}(w)$ for any $w \in \mathcal{A}_n$. In §2.3, we shall describe the left (resp. right) star operations on w in terms of permutations on $\mathbb{Z}$ and in terms of affine matrices when w lies in some $D_L(s_t, s_{t+1})$ (resp. $D_R(s_t, s_{t+1})$), where s_t, s_{t+1} are two Coxeter generators of $\mathcal{A}_n$ with $s_t s_{t+1}$ having order 3. The star operation is one of the most fundamental operations throughout our thesis. Finally, in §2.4, we list some terminology on an affine matrix for later use.

§2.1 THREE DESCRIPTIONS OF THE AFFINE WEYL GROUP $\mathcal{A}_n$

$\mathcal{A}_n$ can be described in the following three different ways.

(i) By generators and relations

$$\mathcal{A}_n = \langle s_i \mid 1 < i < n,\ (s_i s_j)^{m_{ij}} = 1, \text{ for } 1 < j < n \rangle$$

where

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } \bar{i} \neq \bar{j},\ \overline{j \pm 1} \\ 3 & \text{if } \bar{i} = \overline{j \pm 1} \end{cases}$$

with $i \to \bar{i}$ to be the natural map from $\mathbb{Z}$ to the set of the residue classes $\underline{n} = \{\bar{1},\bar{2},\ldots,\bar{n}\}$. We denote $\Delta = \{s_i \mid 1 \leq i \leq n\}$.

(ii) Regarded as a set of permutations on $\mathbb{Z}$.

$$A_n' = \left\{ w:\mathbb{Z} \to \mathbb{Z} \;\middle|\; \begin{array}{l} (i+n)w = (i)w + n \text{ for } i \in \mathbb{Z} \\ \sum_{t=1}^{n} (t)w = \sum_{t=1}^{n} t \end{array} \right\}$$

The relation between these two descriptions is as follows: For any i, $1 \leq i \leq n$, s_i corresponds the permutation

$$t \longrightarrow \begin{cases} t & \text{if } \bar{t} \neq \bar{i}, \overline{i+1} \\ t-1 & \text{if } \bar{t} = \overline{i+1} \\ t+1 & \text{if } \bar{t} = \bar{i} \end{cases}$$

for $t \in \mathbb{Z}$.

(iii) Regarded as the set A_n'' of all $\infty \times \infty$ affine matrices w of type A_{n-1} which are defined as follows:

(a) The integer set $\mathbb{Z}$ is the set parametrising its rows (resp. columns). The integers parametrising its rows (resp. columns) are monotone increasing from top to bottom (resp. from left to right).

(b) The entries in each of its rows (resp. columns) are all zero except for one which is 1.

(c) Let $\{e(u,j_u) \mid u \in \mathbb{Z}\}$ be the set of its non-zero entries, where $e(u,j_u)$ lies in its (u,j_u)-position. Then $j_{u+n} = j_u + n$ for any $u \in \mathbb{Z}$ and $\sum_{u=1}^{n} j_u = \sum_{u=1}^{n} u$.

Clearly, an element $w \in A_n''$ is entirely determined by any of its non-zero entry sets $\{e(u,j_u)\,|\,u \in S\}$ with $S \subset \mathbb{Z}$, $|S| = n$ and $\bar{S} = \{\bar{1},\bar{2},\ldots,\bar{n}\}$, where $\bar{S}$ is the image of S under the natural map $i \to \bar{i}$.

For any $X, Y \in A_n''$ with $\{e_X(i,j)\,|\,i,j \in \mathbb{Z}\}$ and $\{e_Y(i',j')\,|\,i',j' \in \mathbb{Z}\}$ as their entry sets, let $Z = X\cdot Y$ be an $\infty \times \infty$ matrix satisfying condition (a) with its (i,j)-entry $\sum_{k\in\mathbb{Z}} e_X(i,k)e_Y(k,j)$ for any $i,j \in \mathbb{Z}$. It is easily shown that $Z \in A_n''$ and A_n'' is a group with such a multiplication.

For $w \in A_n'$, we define an $\infty \times \infty$ matrix X_w which satisfies condition (a), (b) with $\{e(t,(t)w)\,|\,t \in \mathbb{Z}\}$ as its non-zero entries. Then $X_w \in A_n''$. So we can define a map from A_n' to A_n'' by $w \to X_w$. It is obvious that such a map is a group isomorphism, where s_i, $1 \le i \le n$, corresponds $X_i \in A_n''$ with its non-zero entries $\{e_{X_i}(u,j_u)\,|\,u \in \mathbb{Z}\}$ satisfying

$$j_u = \begin{cases} u & \text{if } \bar{u} \neq \bar{i}, \overline{i+1} \\ u-1 & \text{if } \bar{u} = \overline{i+1} \\ u+1 & \text{if } \bar{u} = \bar{i}. \end{cases}$$

From now on, we shall identify A_n with A_n' and A_n'', and denote them all by A_n. The set $\Delta = \{s_i\,|\,1 \le i \le n\}$ always denotes its (distinguished) Coxeter generators. We stipulate that $s_{i+qn} = s_i$ for any $q, i \in \mathbb{Z}$.

§2.2 <u>THE FUNCTIONS $\ell(w)$, $\mathcal{L}(w)$, $\mathcal{R}(w)$ ON THE AFFINE WEYL GROUP A_n.</u>

Let us list some simple properties of A_n.

<u>Lemma 2.2.1</u> <u>For $y \in A_n$, $s_i \in \Delta$,</u>

(i) <u>$w = s_i \cdot y$ is obtained from y by transposing the $(i+qn)$-th row with the $(i+1+qn)$-th row for every $q \in \mathbb{Z}$.</u>

(ii) <u>$w = y \cdot s_i$ is obtained from y by transposing the $(i+qn)$-th column with the $(i+1+qn)$-th column for every $q \in \mathbb{Z}$.</u>

(iii) <u>$w = y^{-1}$ is obtained from y by transposing y.</u>

<u>Proof</u>: By definition of the affine matrices and their multiplication. □

Let $\ell(\ )$ be the length function of $\tilde{A}_n$ regarded as a Coxeter group. Then we have

<u>Lemma 2.2.2</u> <u>$\ell(y) = \sum_{1<i<j<n} \left|\left[\frac{(j)y-(i)y}{n}\right]\right|$ for $y \in \tilde{A}_n$, where $[h]$ is the integer part of h for $h \in \mathbb{Q}$.</u>

<u>Proof</u>: Let us compare $\sum_{1<i<j<n} \left|\left[\frac{(j)y-(i)y}{n}\right]\right|$ with $\sum_{1<i<j<n} \left|\left[\frac{(j)s_t y-(i)s_t(y)}{n}\right]\right|$ for any $1 < t < n$. Since

$$(h)s_t y = \begin{cases} (h)y & \text{if } \bar{h} \notin \{\bar{t}, \overline{t+1}\} \\ (h+1)y & \text{if } \bar{h} = \bar{t} \\ (h-1)y & \text{if } \bar{h} = \overline{t+1} \end{cases}$$

this implies that

$$\left|\left[\frac{(j)s_t y-(i)s_t y}{n}\right]\right| = \left|\left[\frac{(j)y-(i)y}{n}\right]\right|, \text{ if } \{\bar{i},\bar{j}\} \cap \{\bar{t},\overline{t+1}\} = \emptyset.$$

When $t \neq n$, $\displaystyle\sum_{\substack{1<i<j<n \\ |\{\bar{i},\bar{j}\}\cap\{\bar{t},\overline{t+1}\}|=1}} \left|\left[\frac{(j)y-(i)y}{n}\right]\right| = \sum_{\substack{1<i<j<n \\ |\{\bar{i},\bar{j}\}\cap\{\overline{t+1}\}|=1}} \left|\left[\frac{(j)s_t y-(i)s_t y}{n}\right]\right|$

When $t = n$, $1 < i < n$, we assume that $(n)y - (i)y = kn + r$ and $(i)y - (1)y = k'n + r'$ with $k, k', r, r' \in \mathbb{Z}$ and $0 < r, r' < n-1$ (actually, $1 < r, r' < n-1$ since $\overline{(n)y} \neq \overline{(i)y} \neq \overline{(1)y}$). Then

$$\left|\left[\frac{(n)s_t y-(i)s_t y}{n}\right]\right| = \left|\left[\frac{n+(1)y-(i)y}{n}\right]\right| = \left|\left[\frac{(-k')n+(n-r')}{n}\right]\right| = |-k'| = \left|\left[\frac{(i)y-(1)y}{n}\right]\right|$$

and

$$\left|\left[\frac{(i)s_t y-(1)s_t y}{n}\right]\right| = \left|\left[\frac{(i)y+n-(n)y}{n}\right]\right| = \left|\left[\frac{(-k)n+(n-r)}{n}\right]\right| = |-k| = \left|\left[\frac{(n)y-(i)y}{n}\right]\right|.$$

So we also have $$\sum_{\substack{1<i\leq j\leq n \\ |\{\bar{i},\bar{j}\}\cap\{\bar{1},\bar{n}\}|=1}} \left|\left[\frac{(j)s_n y-(i)s_n y}{n}\right]\right|$$

$$= \sum_{\substack{1<i\leq j\leq n \\ |\{\bar{i},\bar{j}\}\cap\{\bar{1},\bar{n}\}|=1}} \left|\left[\frac{(j)y-(i)y}{n}\right]\right|$$

On the other hand, when $t \neq n$, we assume that $(t+1)y - (t)y = kn + r$ with $k, r \in \mathbb{Z}$ and $1 < r < n$.

Then $\left|\left[\frac{(t+1)s_t y-(t)s_t y}{n}\right]\right| = \left|\left[\frac{(t)y-(t+1)y}{n}\right]\right| = \left|\left[\frac{(-k-1)n+(n-r)}{n}\right]\right|$

$$= |-k-1| = \begin{cases} \left|\left[\frac{(t+1)y-(t)y}{n}\right]\right| + 1 & \text{if } k > 0 \\ \left|\left[\frac{(t+1)y-(t)y}{n}\right]\right| - 1 & \text{if } k < 0 \end{cases} \qquad (1)$$

When $t = n$, we assume that $(n)y-(1)y = hn + u$ with $h, u \in \mathbb{Z}$ and $1 < u < n$. Then

$$|[\frac{(n)s_t y-(1)s_t y}{n}]| = |[\frac{2n+(1)y-(n)y}{n}]| = |[\frac{(1-h)n+(n-u)}{n}]|$$

$$= |1-h| = \begin{cases} |[\frac{(n)y-(1)y}{n}]| + 1 & \text{if } h < 0 \\ |[\frac{(n)y-(1)y}{n}]| - 1 & \text{if } h > 0 \end{cases} \qquad (2)$$

So $$\sum_{1<i<j<n} |[\frac{(j)s_t y-(i)s_t y}{n}]| = \sum_{1<i<j<n} |[\frac{(j)y-(i)y}{n}]| \pm 1 \qquad (3)$$

It is easily seen that $\sum_{1<i<j<n} |[\frac{(j)y-(i)y}{n}]| = 0$ if and only if for any $1 < i < j < n$, the inequality $1 < (j)y - (i)y < n$ holds if and only if $y = 1$. (4)

Then formulae (3) and (4) imply that $\ell(y) > \sum_{1<i<j<n} |[\frac{(j)y-(i)y}{n}]|$.

To show equality, it suffices to show that for any $y \neq 1$, $\exists$ at least one t, $1 < t < n$, such that

$$\sum_{1<i<j<n} |[\frac{(j)s_t y-(i)s_t y}{n}]| = \sum_{1<i<j<n} |[\frac{(j)y-(i)y}{n}]| - 1$$

i.e. to show that at least one of the inequalities

$$(t+1)y - (t)y < 0, \quad \text{for some } t, \ 1 < t < n.$$
$$(n)y - (1)y > n$$

holds, or, equivalently, to show that if $(t+1)y-(t)y > 0$ for all $1 < t < n$, then $(n)y-(1)y > n$. In general, we have $(n)y-(1)y = \sum_{t=1}^{n-1} ((t+1)y-(t)y) > n-1$. If $(n)y-(1)y = n-1$, then $(t+1)y-(t)y = 1$ for any t, $1 < t < n$. Hence it follows from the equation $\sum_{t=1}^{n} t = \sum_{t=1}^{n} (t)y = n\cdot(1)y + \sum_{t=1}^{n-1} t$ that $(1)y = 1$ and so $(t)y = t$ for all t, $1 \leq t \leq n$. But this means $y = 1$. It contradicts $y \neq 1$. Also, since $\overline{(n)y} \neq \overline{(1)y}$, we have $(n)y-(1)y \neq n$. Thus it follows that $(n)y - (1)y > n$. Our assertion is proved. □

Corollary 2.2.3 (of the proof of Lemma 2.2.2)

Let $w,y \in \hat{A}_n$, $s_t \in \Delta$ with $w = s_t y$. Then

(i) $\ell(w) = \ell(y) + 1 \iff (t+1)y > (t)y$

(ii) $\ell(w) = \ell(y) - 1 \iff (t+1)y < (t)y$

Proof: We write $(t+1)y - (t)y = kn + r$ with $k, r \in \mathbb{Z}$ and $1 \leq r < n$ when $t \neq n$. Then by formula (1) and Lemma 2.2.2, $\ell(w) = \ell(y) + 1 \iff k \geq 0 \iff (t+1)y-(t)y > 0 \iff (t+1)y > (t)y$.

Also, we write $(n)y-(1)y = hn + u$ with $h, u \in \mathbb{Z}$ and $1 \leq u < n$ when $t = n$. Then by formula (2) and Lemma 2.2.2, $\ell(w) = \ell(y) + 1 \iff h \leq 0 \iff (n)y-(1)y < n \iff (n+1)y-(n)y > 0 \iff (n+1)y > (n)y$.

So (i) follows. Since (ii) is equivalent to (i), our proof is complete. □

Corollary 2.2.3 can be restated in terms of matrices as follows:

Corollary 2.2.3': If w is obtained from y by transposing the $(i+qn)$-th row with the $(i+1+qn)$-th row for all $q \in \mathbb{Z}$, and if $e_y(u,j_u)$ is the non-zero entry of y lying in the (u,j_u)-position for any $u \in \mathbb{Z}$, then $\ell(w) = \ell(y) \pm 1$ and

(i) $\ell(w) = \ell(y) + 1 \Longleftrightarrow j_{i+1} > j_i$

(ii) $\ell(w) = \ell(y) - 1 \Longleftrightarrow j_{i+1} < j_i$ □

The following two lemmas concern the functions $\mathcal{L}(\ )$, $\mathcal{R}(\ )$ on $\tilde{A}_n$.

Lemma 2.2.4 For any $w \in \tilde{A}_n$, we have

$$\mathcal{L}(w) = \{s_t \in \Delta \mid (t+1)w < (t)w\}$$

$$\mathcal{R}(w) = \{s_t \in \Delta \mid (t+1)w^{-1} < (t)w^{-1}\}$$

Proof: By definition of the functions $\mathcal{L}(\ )$, $\mathcal{R}(\ )$ (see Chapter 1) and Corollary 2.2.3. □

Lemma 2.2.5 For any $w \in \tilde{A}_n$, we have $0 \le |\mathcal{L}(w)| < n$ and $0 \le |\mathcal{R}(w)| < n$. Moreover, the following three statements are equivalent:

(i) $|\mathcal{L}(w)| = 0$ (ii) $|\mathcal{R}(w)| = 0$ (iii) $w = 1$

Proof: Obviously, $w = 1$ implies $|\mathcal{L}(w)| = |\mathcal{R}(w)| = 0$.

In the proof of Lemma 2.2.2, we have shown that $w \neq 1$ implies $|\mathcal{L}(w)| > 0$. And so $w \neq 1$ implies $w^{-1} \neq 1$ and then $|\mathcal{L}(w^{-1})| > 0$. It turns out $|\mathcal{R}(w)| > 0$. Therefore the latter part of the lemma has been proved. Now it is enough to show that $|\mathcal{L}(w)| \neq n$. Otherwise, we would have $(t)w > (t+1)w$ for all $1 \le t \le n$. So by Lemma 2.2.4,

$$(1)w > (2)w > \ldots > (n)w > (n+1)w = (1)w + n, \text{ i.e. } n < 0.$$

This is impossible. Our result follows. □

2.3 THE SUBSETS $\mathcal{D}_L(s_t)$, $\mathcal{D}_R(s_t)$ OF THE AFFINE WEYL GROUP A_n, $n > 3$.

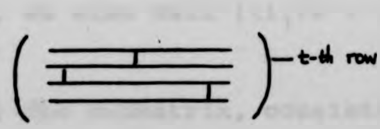
Now we assume $n > 3$. Let us denote $\mathcal{D}_L(s_t, s_{t+1})$ and $\mathcal{D}_R(s_t, s_{t+1})$ by $\mathcal{D}_L(s_t)$ and $\mathcal{D}_R(s_t)$, respectively, for any $t \in \mathbb{Z}$. We call the map $w \to {}^*w$ in $\mathcal{D}_L(s_t)$ the left star operator on w and $y \to y^*$ in $\mathcal{D}_R(s_t)$ the right star operator on y. In terms of permutations on $\mathbb{Z}$, $w \in A_n$ lies in $\mathcal{D}_L(s_t)$ if and only if w satisfies one of the following inequalities:

(i) $(t+1)w < (t)w < (t+2)w$ (ii) $(t+1)w < (t+2)w < (t)w$

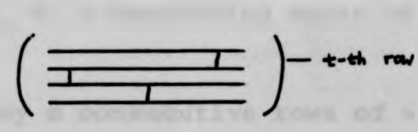
(iii) $(t)w < (t+2)w < (t+1)w$ (iv) $(t+2)w < (t)w < (t+1)w$.

Also, in terms of matrices, $w \in A_n$ lies in $\mathcal{D}_L(s_t)$ if and only if w has one of the following forms:

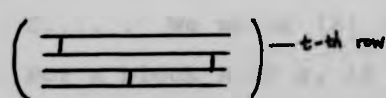
(i')



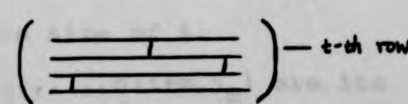
(ii')



(iii')



(iv')



where form (α') is the matrix version of inequality (α) for α = i, ii, iii, iv.

If $w \in \mathcal{D}_L(s_t)$, then w, ${}^*w$ (or ${}^*w$, w) in $\mathcal{D}_L(s_t)$ are either of forms (i'), (ii'), or of forms (iii'), (iv'), respectively.

Since $\mathcal{D}_R(s_t) = \{w \in A_n \mid w^T \in \mathcal{D}_L(s_t)\}$ by Lemma 2.2.1(iii), we have the corresponding results on $\mathcal{D}_R(s_t)$, where w^T is the transpose of the matrix w.

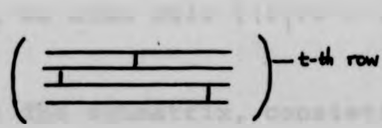
2.3 THE SUBSETS $\mathcal{D}_L(s_t)$, $\mathcal{D}_R(s_t)$ OF THE AFFINE WEYL GROUP $\tilde{A}_n$, $n > 3$.

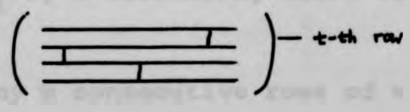
Now we assume $n > 3$. Let us denote $\mathcal{D}_L(s_t, s_{t+1})$ and $\mathcal{D}_R(s_t, s_{t+1})$ by $\mathcal{D}_L(s_t)$ and $\mathcal{D}_R(s_t)$, respectively, for any $t \in \mathbb{Z}$. We call the map $w \to {}^*w$ in $\mathcal{D}_L(s_t)$ the left star operator on w and $y \to y^*$ in $\mathcal{D}_R(s_t)$ the right star operator on y. In terms of permutations on $\mathbb{Z}$, $w \in \tilde{A}_n$ lies in $\mathcal{D}_L(s_t)$ if and only if w satisfies one of the following inequalities:

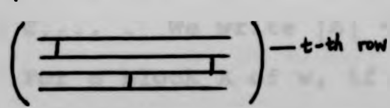
(i) $(t+1)w < (t)w < (t+2)w$ (ii) $(t+1)w < (t+2)w < (t)w$

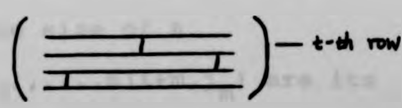
(iii) $(t)w < (t+2)w < (t+1)w$ (iv) $(t+2)w < (t)w < (t+1)w$.

Also, in terms of matrices, $w \in \tilde{A}_n$ lies in $\mathcal{D}_L(s_t)$ if and only if w has one of the following forms:

(i')


(ii')


(iii')


(iv')


where form (α') is the matrix version of inequality (α) for α = i, ii, iii, iv.

If $w \in \mathcal{D}_L(s_t)$, then w, *w (or *w, w) in $\mathcal{D}_L(s_t)$ are either of forms (i'), (ii'), or of forms (iii'), (iv'), respectively.

Since $\mathcal{D}_R(s_t) = \{w \in \tilde{A}_n \mid w^T \in \mathcal{D}_L(s_t)\}$ by Lemma 2.2.1(iii), we have the corresponding results on $\mathcal{D}_R(s_t)$, where w^T is the transpose of the matrix w.

§2.4 SOME TERMINOLOGY

Now we make some conventions for later use. Fix an element $w \in A_n$.

(i) When we say "an entry of w", we always mean that this entry is non-zero unless the contrary is indicated. When we mention "an entry of w" (not necessarily non-zero), it means that we know both its value and its position in w. The entries are usually denoted by e in expressions such as $e(w)$, $e(i,j)$, $e_{ij}(i,j)$, $e(i,(w))$, $e(w),j)$, etc, if it lies in the (i,j)-position of w.

(ii) Suppose that $E = \{e(i_1,j_1),\dots,e(i_t,j_t)\}$ is a subset of entries of w such that $i_1 < \dots < i_t$ and $j_1 > \dots > j_t$. Then E is called a descending chain of entries of w (briefly, a descending chain of w). Let $|E| = t$ be its size. In that case, we also call $\{(i_1)w > \dots > (i_t)w\}$ a descending chain of w.

(iii) The submatrix, consisting of any m consecutive rows of w with $m < n$, is called a block. Blocks will be written as A, B, C,... . We write $|A| = m$ as the size of A.

(iv) For a block A of w, if $e(i+1,j_1),\dots,e(i+m,j_m)$ are its entries such that $i \in \mathbb{Z}$ and $j_1 > \dots > j_m$, then A is called a block of w whose entry set is a descending chain (briefly, a DC block of w). In that case, if the entries $e(i,j_o)$, $e(i+m+1, j_{m+1})$ of w satisfy $j_o < j_1$ and $j_m < j_{m+1}$, then A is also called a maximal DC block of w (briefly, an MDC block of w). If A is contained in a block B such that $j_o < j_1$ and $j_m < j_{m+1}$ whenever $e(i,j_o)$, $e(i+m+1, j_{m+1})$ lie in B, then A is called a local MDC block of w in B. Clearly, any MDC block is also a local MDC block and any DC block is a local MDC block in itself.

Assume that A, B are two blocks of w. Then $A \cup B$ denotes the union of A and B. In particular, when A, B are consecutive blocks of w, we see that $[A,B] = \binom{A}{B}$ is a block of w, provided that $|A| + |B| < n$.

(v) Let e, e' be two entries (not necessarily non-zero) of w. Let f, f' (resp. g, g') be two rows (resp. columns) of w. We define

$$r(e,e') = \begin{cases} 0 & \text{if } e, e' \text{ lie in the same row of } w \\ \pm m & \text{otherwise} \end{cases}$$

where $m-1 \geq 0$ is the number of rows between e and e'. $r(e,e') = m$ (resp. $r(e,e') = -m$) if e' is below (resp. above) e.

We also define

$$c(e,e') = \begin{cases} 0 & \text{if } e, e' \text{ lie in the same column of } w \\ \pm \ell & \text{otherwise} \end{cases}$$

where $\ell-1 \geq 0$ is the number of columns between e and e'. $c(e,e') = \ell$ (resp. $c(e,e') = -\ell$) if e' is on the right (resp. left) of e.

Similarly, we can define $r(f,f')$, $c(g,g')$. Clearly, $r(e,e') = -r(e',e)$, $c(e,e') = -c(e',e)$, $r(e,e'') = r(e,e') + r(e',e'')$, $c(e,e'') = c(e,e') + c(e',e'')$ for any entry e" of w (not necessarily non-zero), etc.

(vi) For any entries e, e' of w, we say that e' is congruent to e if $r(e,e') = c(e,e') \in n\mathbb{Z}$.

For any rows f, f' (resp. columns g, g') of w, we say that f' is congruent to f (resp. g' is congruent to g) if $r(f,f') \in n\mathbb{Z}$ (resp. $c(g,g') \in n\mathbb{Z}$).

The set of all entries (resp. rows, columns) congruent to a certain entry (resp. row, column) of w is called an entry class of w (resp. a row class, a column class). Let $\bar{e}$ be a set of m entry classes of w. Then we define the size of $\bar{e}$ by $|\bar{e}| = m$.

Let $\bar{e}$ be a set of entry classes of w. Let ζ be the set of all rows and columns of w each of which contains some entry of $\bar{e}$. Then ζ is called a set of row-column classes (briefly, rc-classes) of w. We define the size of ζ, written $|\zeta|$, by the size of $\bar{e}$.

We call two blocks A, B of w congruent if the top row of B is congruent to the top row of A and $|A| = |B|$. For any block A of w with $|A| = m$, we usually denote any of its congruent blocks also by A. Moreover, let $S_A = \{e(i+u,(w)) \mid 1 < u < m\}$ be the set of entries contained in A, $S_A(q) = \{e(i+u+qn,(w)) \mid 1 < u < m\}$ for some $q \in \mathbb{Z}$. Then by abuse of terminology, we also call $S_A(q)$ a block of w and denote it by A.

(vii) For any $w \in \mathbb{A}_n$, we say that w has the form $(A_\ell,\dots,A_1)$ at i if w has the form

$$\begin{pmatrix} \vdots \\ A_\ell(w) \\ \vdots \\ A_1(w) \\ \vdots \end{pmatrix}, \text{ where } A_t(w),\ 1 < t < \ell, \text{ are consecutive blocks of}$$

w with $|A_t(w)| = m_t$ and $\sum_{t=1}^{\ell} m_t < n$. $i+1$ is the integer labelling the first row of $A_\ell(w)$ in w. If we first say that w has the form

$(A_\ell,\ldots,A_1)$ at i and subsequently mention "the u-th entry of $A_t(w)$", it always means that this $A_t(w)$ is a single block of w lying between the (i+1)-th row and the (i+n)-th row. We denote the u-th entry of $A_t(w)$ by $e((w), j_t^u(w))$, $e((w), j_t^u(w,i))$ or $e((w), j_{A_t}^u(w))$. If all $A_1(w),\ldots,A_\ell(w)$ are DC (resp. MDC) blocks, then we say that w has the DC (resp. MDC) form $(A_\ell,\ldots,A_1)$ at i. If all $A_1(w),\ldots,A_\ell(w)$ are local MDC blocks in the block $[A_\ell,\ldots,A_1]$, then we say that w has a local MDC form $(A_\ell,\ldots,A_1)$ at i. Sometimes, for the sake of emphasizing that the block $A_t(w)$ of w lies between the (i+1)-th rows and the (i+n)-th row, we denote $A_t(w)$ by $A_t(w,i)$.

(viii) In the present thesis, for any $w \in A_n$, when we say "transposing the i-th and the j-th rows (resp. columns) of w", it always means that we transpose the (i+qn)-th and the (j + qn)-th rows (resp. columns) of w for every $q \in \mathbb{Z}$. We make the same conventions for the transformations on the blocks of w. We also make similar conventions on $\tilde{w} \in \tilde{A}_n$. (The notation $\tilde{A}_n$ will be introduced later).

CHAPTER 3 : THE PARTITION OF n ASSOCIATED WITH AN ELEMENT OF THE AFFINE WEYL GROUP $\tilde{A}_n$

Let Λ_n be the set of partitions of n, $n > 2$. Let $\underline{\Delta}$ be the set of proper subsets of Δ. We shall define two maps: $\sigma: \tilde{A}_n \to \Lambda_n$ and $\pi: \underline{\Delta} \to \Lambda_n$, and show that the fibre $\sigma^{-1}(\lambda)$ is invariant under the star operations and the inverse for any $\lambda \in \Lambda_n$. These two maps, especially the first one, are at the heart of our thesis. In this chapter, we shall also discuss some relations between these two maps.

For any $J \in \underline{\Delta}$, W_J is by definition a standard parabolic subgroup of $\tilde{A}_n$ generated by J.

<u>Definition 3.1</u> A map $\pi: \underline{\Delta} \to \Lambda_n$ is defined as follows: For $J = J_1 \cup \ldots \cup J_r \in \underline{\Delta}$ with $W_J = W_{J_1} \times \ldots \times W_{J_r}$ and W_{J_j} indecomposable, $1 < j < r$, and $|J_1| > \ldots . > |J_r|$, we define
$\pi(J) = \{|J_1| + 1 > |J_2| + 1 > \ldots > |J_r| + 1 > 1 > \ldots > 1\} \in \Lambda_n$.

It is easily seen that the map π is well defined and surjective.

<u>Definition 3.2</u> A map $\sigma: \tilde{A}_n \to \Lambda_n$ is defined as follows:
To $w \in \tilde{A}_n$, we associate a sequence of integers $d_1 < d_2 < \ldots < d_r = n$ as follows: d_k is the maximum cardinal of a subset of $\mathbb{Z}$ whose elements are incogruent to each other mod n and which is a disjoint union of k subsets each of which has its natural order reversed by w. Let $\lambda_1 = d_1$, $\lambda_j = d_j - d_{j-1}$ for $1 < j < r$. Then it is clear that $\lambda_1 > \lambda_2 > \ldots > \lambda_r$ and $\sum_{j=1}^{r} \lambda_j = n$. We define $\sigma(w) = \{\lambda_1 > \lambda_2 > \ldots > \lambda_r\}$.

The integers d_k, $1 \leq k \leq r$, in Definition 3.2, can also be described in terms of an affine matrix: d_k is the maximum cardinal of a subset of entries of w whose elements are incongruent to each other and which is a disjoint union of k descending chains of w.

The following lemma shows that for any $\lambda \in \Lambda_n$, the fibre $\sigma^{-1}(\lambda)$ is invariant under the inverse.

<u>Lemma 3.3</u> <u>$\sigma(w) = \sigma(w^{-1})$ for any $w \in \tilde{A}_n$</u>

<u>Proof</u>: We know from Lemma 2.2.1 (iii) that the matrix w^{-1} is the transpose of w. Since the operation of transposing a matrix keeps any descending chain and sends any two incongruent entries to incongruent ones, our result then follows immediately from the definition of the map σ. □

For $w \in \tilde{A}_n$, the following condition on $S = S_1 \dot{\cup} \ldots \dot{\cup} S_t \subset \mathbb{Z}$ is called $C_n(w,t)$: Elements of S are incongruent mod n and $(a)w > (b)w$ in S_j, $1 \leq j \leq t$, implies $a < b$. Let $E = E_1 \cup \ldots \cup E_t$ be the subset of entries of w with $E_j = \{e(a,(a)w) \mid (a)w \in S_j\}$, $1 \leq j \leq t$. Then condition $C_n(w,t)$ on $S = S_1 \cup \ldots \cup S_t$ is equivalent to the following condition on $E = E_1 \cup \ldots \cup E_t$: Elements of E are incongruent and E_j is a descending chain for any $1 \leq j \leq t$. So we can also say that $E = E_1 \cup \ldots \cup E_t$ satisfies $C_n(w,t)$ and regard condition $C_n(w,t)$ on $E = E_1 \cup \ldots \cup E_t$ as a matrix version of condition $C_n(w,t)$ on $S = S_1 \cup \ldots \cup S_t$.

Now we define a preorder $\geq$ on Λ_n as follows: Let $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r\}$, $\mu = \{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m\} \in \Lambda_n$. We say $\lambda \geq \mu$ if $\lambda_1 + \lambda_2 + \ldots + \lambda_k \geq \mu_1 + \mu_2 + \ldots + \mu_k$ for any $k \geq 1$ with the convention that $\lambda_i = \mu_j = 0$ for all $i > r$ and $j > m$, or, equivalently, if $\lambda_1 + \lambda_2 + \ldots + \lambda_k \geq \mu_1 + \mu_2 + \ldots + \mu_k$ for any $1 \leq k \leq r$. Clearly, this is a partial order on Λ_n. $\lambda \geq \mu$ implies $r \leq m$. For any $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r\} \in \Lambda_n$, we call $\mu = \{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m\} \in \Lambda_n$ the dual partition of λ if μ_k is the number of parts λ_j of λ with $1 \leq j \leq r$ and $\lambda_j \geq k$ for any $1 \leq k \leq m$. The map which sends any element of Λ_n to its dual is an involution of Λ_n and is order-reversing with respect to $\geq$.

Lemma 3.4 For any $w, w' \in A_n$ and $s_\alpha \in \Delta$ with $w' = s_\alpha w$ and $\ell(w') = \ell(w) + 1$, we have $\sigma(w') \geq \sigma(w)$.

Proof: If $S = S_1 \cup \ldots \cup S_t \subset \mathbb{Z}$ satisfies $C_n(w,t)$ for any $t \geq 1$, then $S' = (S_1)w^{-1}w' \cup \ldots \cup (S_t)w^{-1}w'$ satisfies $C_n(w',t)$. This implies that the integer d_t in Definition 3.2 for w' is not less than that for w, for any $t \geq 1$. So $\sigma(w') \geq \sigma(w)$. □

Corollary 3.5 If $w' = ws_\alpha$ in A_n with $s_\alpha \in \Delta$ and $\ell(w') = \ell(w) + 1$, then $\sigma(w') \geq \sigma(w)$.

Proof: This follows immediately from Lemmas 3.3, 3.4. □

The following lemma gives a relation between two maps σ and π on a given element $w \in A_n$.

Lemma 3.6 For any $w \in A_n$, $\pi(\mathcal{L}(w))$, $\pi(\mathcal{R}(w)) \leq \sigma(w)$.

Proof: By Lemma 3.3, it suffices to show that $\pi(\mathcal{L}(w)) \leq \sigma(w)$. Assume that $\mathcal{L}(w) = J = J_1 \cup \ldots \cup J_r \in \underline{\Delta}$ such that $W_J = W_{J_1} \times \ldots \times W_{J_r}$ and W_{J_j} is indecomposable, $1 \leq j \leq r$, with $J_j = \{s_{i_j+1}, s_{i_j+2}, \ldots, s_{i_j+m_j}\}$, $m_1 \geq m_2 \geq \ldots \geq m_r$.

Then $\pi(\mathcal{L}(w)) = \{m_1+1 > m_2+1 > \ldots > m_r+1 > 1 > \ldots > 1\} \in \Lambda_n$.
On the other hand, for t with $1 < t < r$, let

$$S_j = \{(i_j+1)w, (i_j+2)w, \ldots, (i_j + m_j+1)w\}, \quad 1 < j < t.$$

By Lemma 2.2.4, we have

$$(i_j+1)w > (i_j+2)w > \ldots > (i_j + m_j + 1)w, \quad 1 < j < t$$

and this implies that $S = S_1 \cup \ldots \cup S_t$ satisfies $C_n(w,t)$ with $|S| = \sum_{j=1}^{t} (m_j+1)$. So $\sum_{j=1}^{t} (m_j+1) < \sum_{j=1}^{t} \lambda_j$ for any t with $1 < t < \min\{r,h\}$, where we assume $\sigma(w) = \{\lambda_1 > \ldots > \lambda_n\}$.
But this easily implies that $\sum_{j=1}^{t} (m_j+1) < \sum_{j=1}^{t} \lambda_j$ for any $t > 1$.

Therefore $\pi(\mathcal{L}(w)) < \sigma(w)$. □

Now we shall show that for any $\lambda \in \Lambda_n$, the fibre $\sigma^{-1}(\lambda)$ is invariant under the star operations.

<u>Lemma 3.7</u> <u>Assume that $w' = {}^*w$ in $\mathcal{D}_L(\mathscr{S}_i)$ for some $i \in \mathbb{Z}$</u>

<u>Then $\sigma(w') = \sigma(w)$</u>.

<u>Proof:</u> By symmetry, it suffices to discuss the case that $(i+1)w < (i+2)w < (i)w$. In that case, $w' = s_i w$, $\ell(w') = \ell(w)-1$ and then for any $j \in \mathbb{Z}$, we have

$$(j)w' = \begin{cases} (j)w & \text{if } \bar{j} \neq \bar{i}, \overline{i+1} \\ (j+1)w & \text{if } \bar{j} = \bar{i} \\ (j-1)w & \text{if } \bar{j} = \overline{i+1} \end{cases}$$

By Lemma 3.4, it follows that $\sigma(w) > \sigma(w')$. We now need only show that $\sigma(w') > \sigma(w)$.

Let $S = S_1 \cup \ldots \cup S_t \subset \mathbb{Z}$ satisfy $C_n(w,t)$. If $\nexists\ 1 < j < t$, such that S_j contains two elements $(h_1)w$, $(h_2)w$ with $\bar{h}_1 = \bar{1}$ and $h_2 = h_1 + 1$, then $(S)w^{-1}w' = (S_1)w^{-1}w' \cup \ldots \cup (S_t)w^{-1}w'$ satisfies $C_n(w',t)$. Clearly, $|(S)w^{-1}w'| = |S|$. Now suppose that $\exists\ 1 < j < t$ such that S_j contains $(h_1)w$, $(h_2)w$ with $\bar{h}_1 = \bar{1}$ and $h_2 = h_1 + 1$. If $\nexists\ (g)w \in S$ such that $\bar{g} = \overline{1+2}$, then let

$$S'_\ell = \begin{cases} (S_\ell)w^{-1}w' & \text{if } \ell \neq j \\ ((h_1+2)w \cup S_j)w^{-1}w' - (h_1)w' & \text{if } \ell = j \end{cases}$$

where for any sets X,Y, $X-Y$ denotes their set-theoretical difference. If $\exists (g)w \in S$ such that $\bar{g} = \overline{1+2}$, say $(g)w \in S_k$, then it is clear that $k \neq j$. Assume that

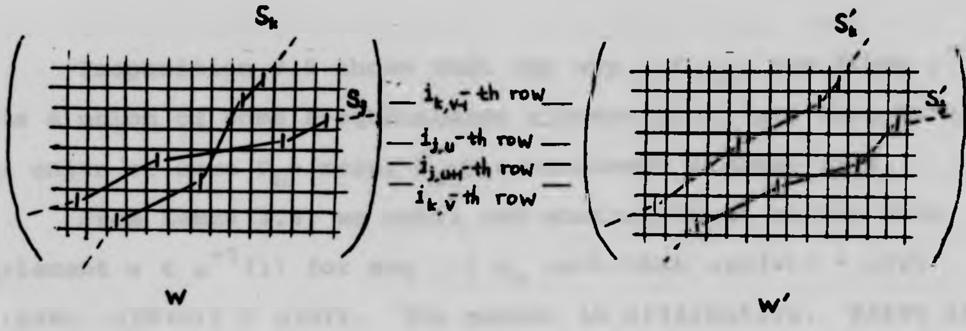
$$S_j = \{(i_{j1})w > (i_{j2})w > \ldots > (i_{j\alpha_j})w \mid i_{j1} < i_{j2} < \ldots < i_{j\alpha_j}\}$$

$$S_k = \{(i_{k1})w > (i_{k2})w > \ldots > (i_{k\alpha_k})w \mid i_{k1} < i_{k2} < \ldots < i_{k\alpha_k}\}$$

and for some $1 < u, u' < \alpha_j$, $1 < v < \alpha_k$, we have $\bar{i}_{ju} = \bar{1}$, $\bar{i}_{kv} = \overline{1+2}$ and $i_{ju'} = i_{ju}+1$. Then $u' = u+1$. By proper choice of S_k, we may assume that $i_{kv} = i_{ju} + 2$. Then let

$$S'_\ell = \begin{cases} (S_\ell)w^{-1}w' & \text{if } \ell \neq k,j \\ \{(i_{j1})w' > \ldots > (i_{j,u-1})w' > (i_{j,u+1})w' > (i_{kv})w' > (i_{k,v+1})w' \\ \quad > \ldots > (i_{k\alpha_k})w'\} & \text{if } \ell = j \\ \{(i_{k1})w' > \ldots > (i_{k,v-1})w' > (i_{j,u})w' > (i_{j,u+2})w' > (i_{j,u+3})w' \\ \quad > \ldots > (i_{j\alpha_j})w'\} & \text{if } \ell = k. \end{cases}$$

An example of this construction of S'_ℓ in terms of matrix is given in the following figure.



where $i_{j,u} = i_{j,u+1}-1 = i_{k,v}-2$. w' is obtained from w by transposing the $i_{j,u}$-th row with the $i_{j,u+1}$-th row.

In both cases, $S' = S_1' \cup \ldots \cup S_t'$ satisfies $C_n(w',t)$ and $|S'| = |S|$. Now for any $S = S_1 \cup \ldots \cup S_t \subset \mathbb{Z}$, $t > 1$, satisfying $C_n(w,t)$, we can find $S' = S_1' \cup \ldots \cup S_t' \subset \mathbb{Z}$ which satisfies $C_n(w',t)$ with $|S'|$ not less than $|S|$. So by the same argument as that in the proof of Lemma 3.4, we have $\sigma(w') > \sigma(w)$. Therefore $\sigma(w) = \sigma(w')$ as required. □

<u>Corollary 3.8</u> <u>If $w' = w^*$ in $\mathcal{D}_R(s_i)$ for some $i \in \mathbb{Z}$, then $\sigma(w') = \sigma(w)$.</u>

<u>Proof</u>: Since $w' = w^*$ in $\mathcal{D}_R(s_i)$, it implies that $w'^{-1} = {}^*(w^{-1})$ in $\mathcal{D}_L(s_i)$. By Lemma 3.7, $\sigma(w'^{-1}) = \sigma(w^{-1})$. So $\sigma(w') = \sigma(w)$ by Lemma 3.3. □

<u>Proposition 3.9</u> <u>If $w, w' \in \mathcal{A}_n$ satisfy $w \underset{P}{\sim} w'$, then $\sigma(w) = \sigma(w')$.</u>

<u>Proof</u>: Since P-equivalence is generated by $w \underset{P_L}{\sim} {}^*w$ in $\mathcal{D}_L(s_i)$ and $y \underset{P_R}{\sim} y^*$ in $\mathcal{D}_R(s_j)$, where s_i, s_j run over Δ, our result follows from Lemma 3.7 and Corollary 3.8. □

Proposition 3.9 shows that for any $\lambda \in \Lambda_n$, the fibre $\sigma^{-1}(\lambda)$ is a union of some P-equivalence classes of $\mathring{A}_n$ and then is also a union of some P_L-(resp. P_R-) equivalence classes of $\mathring{A}_n$.

From Lemma 3.6, we shall ask whether there exists some element $w \in \sigma^{-1}(\lambda)$ for any $\lambda \in \Lambda_n$ such that $\pi(\mathcal{L}(w)) = \sigma(w)$ (resp. $\pi(\mathcal{R}(w)) = \sigma(w)$). The answer is affirmative. First let us show the following result.

Lemma 3.10 For $w \in \mathring{A}_n$, we assume that $E = \{e(i_t + qn, j_t + qn) \mid 1 \leq t \leq \ell, q \in \mathbb{Z}\}$ is a set of entry classes of w such that $i_1 - n < i_\ell < i_{\ell-1} < \ldots < i_1$ and $j_1 - n < j_\ell < j_{\ell-1} < \ldots < j_1$. Let $S = \{e(i_u', j_u') \mid 1 \leq u \leq m\}$ be a descending chain of w with $i_m' < \ldots < i_1'$ and $j_m' > \ldots > j_1'$. Then $|E \cap S| \leq 1$.

Proof: Otherwise, there exist α, β with $1 \leq \alpha < \beta \leq m$ such that $\bar{j}_\alpha' = \bar{j}_a$ and $\bar{j}_\beta' = \bar{j}_b$. Then there exist $p, q \in \mathbb{Z}$ such that $j_\alpha' = j_a + pn$, $i_\alpha' = i_a + pn$, $j_\beta' = j_b + qn$ and $i_\beta' = i_b + qn$. So the inequality $(i_a - i_b) + (p-q)n < 0$ follows from $i_\alpha' - i_\beta' < 0$ and $(j_a - j_b) + (p-q)n > 0$ from $j_\alpha' - j_\beta' > 0$, i.e. We have

$$\begin{cases} i_a - i_b < (q-p)n & (1) \\ j_a - j_b > (q-p)n & (2) \end{cases}$$

Since $0 < |i_a - i_b|, |j_a - j_b| < n$, the inequalities (1), (2) imply $q-p \geq 0$ and $q-p \leq 0$, respectively. Hence $q = p$ and we have

$$\begin{cases} i_a - i_b < 0 & (3) \\ j_a - j_b > 0 & (4) \end{cases}$$

But the inequality (3) implies $a > b$ and (4) implies $b > a$. This gives a contradiction. So our result follows. □

Fix a partition $\lambda = \{\lambda_1 > \lambda_2 > \ldots > \lambda_r\} \in \Lambda_n$. Let $\lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_r}$ be a permutation of $\lambda_1, \lambda_2, \ldots, \lambda_r$. Assume that $J = J_1 \cup \ldots \cup J_r \in \underline{\Delta}$ with $J_j = \{s_{\alpha_j+1}, s_{\alpha_j+2}, \ldots, s_{\alpha_j+a_j-1}\}, 1 < j < r$, and $\alpha_j = i + \sum_{h=j+1}^{r} \lambda_{a_h}$ for some $i \in \mathbb{Z}$. Let $w_o{}^J$ be the longest element in W_J. Then $\pi(\mathcal{L}(w_o{}^J)) = \pi(\mathcal{R}(w_o{}^J)) = \pi(J) = \lambda$.

As an affine matrix, $w_o{}^J$ has the form

$$\begin{pmatrix} \ddots & & & & \\ & K_r & & & \\ & & K_{r-1} & & \\ & & & \ddots & \\ & & & & K_1 \\ & & & & & \ddots \end{pmatrix}$$

where $K_j = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ is a $\lambda_{a_j} \times \lambda_{a_j}$ diagonal block of $w_o{}^J$, $1 < j < r$. The integer labelling the first row (resp. column) of K_j in $w_o{}^J$ is α_j+1. In other words, $w_o{}^J$ has the MDC form $(A_r, A_{r-1}, \ldots, A_1)$ at 0 with the t-th entry $e(i_h^t, j_h^t)$ of $A_h(w_o{}^J)$, $1 < h < r$, $1 < t < \lambda_{a_h}$, where $i_h^t = \alpha_h + t$, $j_h^t = \alpha_{h-1} + 1 - t$.

Assume that $\mu = \{\mu_1 > \mu_2 > \ldots > \mu_{\lambda_1}\}$ is the dual partition of λ. Then it is clear that for any $1 < t < \lambda_1$,

$$i^t_{a_{\beta_1(t)}} - n < i^t_{a_{\beta_{\mu_t}(t)}} < i^t_{a_{\beta_{\mu_t-1}(t)}} < \ldots < i_{a_{\beta_1(t)}} \quad \text{and}$$

$$j^t_{a_{\beta_1(t)}} - n < j^t_{a_{\beta_{\mu_t}(t)}} < j^t_{a_{\beta_{\mu_t-1}(t)}} < \ldots < j^t_{a_{\beta_1(t)}} \quad \text{where}$$

$\beta_1(t), \beta_2(t), \ldots, \beta_{\mu_t}(t)$ is the subsequence of $1,2,\ldots,r$ such that for any h with $1 < h < \mu_t$, we have $\lambda_{a_{\beta_h(t)}} > t$. Let

$E_t = \{e(i^t_{a_{\beta_h}(t)} + qn,\ j^t_{a_{\beta_h}(t)} + qn) \mid 1 < h < \mu_t,\ q \in \mathbb{Z}\}$ be the set of entry classes of w for $1 < t < \lambda_1$. Then by Lemma 3.10, the intersection of E_t with any descending chain of w_o^J has cardinal at most 1. Assume that $S = S_1 \cup \ldots \cup S_k$ is a disjoint union of k descending chains of w_o^J satisfying $C_n(w,k)$, $k > 1$. Then the cardinal of the intersection of E_t with S is at most $\min\{\mu_t, k\}$, for any t with $1 < t < \lambda_1$. Since $\bigcup_{t=1}^{\lambda_1} E_t$ is the full set of entry classes of w_o^J, it implies that $|S| < \sum_{t=1}^{\lambda_1} \min\{\mu_t, k\} = \sum_{j=1}^{k} \lambda_j$. Hence $\sigma(w_o^J) < \lambda$. By Lemma 3.6, this implies that $\sigma(w_o^J) = \pi(\mathcal{L}(w_o^J)) = \pi(\mathcal{R}(w_o^J)) = \lambda$. So we have

<u>Lemma 3.11</u> <u>For any $J \in \underline{\Delta}$, let w_o^J be the longest element in W_J. Then we have</u>

$$\underline{\sigma(w_o^J) = \pi(\mathcal{L}(w_o^J)) = \pi(\mathcal{R}(w_o^J)) = \pi(J).} \qquad \square$$

<u>Corollary 3.12</u> <u>The map $\sigma: \mathcal{A}_n \to \Lambda_n$ is surjective.</u>

<u>Proof</u>: Since the map $\pi: \underline{\Delta} \to \Lambda_n$ is surjective, and since by Lemma 3.11, for any $J \in \underline{\Delta}$, there exists an element $w \in \mathcal{A}_n$ such that $\sigma(w) = \pi(J)$, our conclusion follows immediately. $\square$

CHAPTER 4 : SOME CELLS OF THE AFFINE WEYL GROUP $\tilde{A}_n$

Let $C_t = \{s_r s_{r+1} \ldots s_t;\ s_{r'} s_{r'-1} \ldots s_t \mid r < t,\ r' > t\}$ for $1 < t < n$ and let $C = \bigcup_{t=1}^{n} C_t$. In this chapter, we shall first show that the sets $\{1\}$, C are both 2-sided cells of $\tilde{A}_n$ corresponding to the partitions $\{1 > 1 > \ldots > 1\}$, $\{2 > 1 > \ldots > 1\} \in \Lambda_n$, respectively. We shall also show that C_t is a left cell of $\tilde{A}_n$ for any t, $1 < t < n$. Then by applying this result to $\tilde{A}_2$, we can find all cells of $\tilde{A}_2$.

<u>Proposition 4.1</u> <u>Let W be any Coxeter group. Then the set consisting of the identity element 1 is a 2-sided cell of W.</u>

<u>Proof</u>: Otherwise, there exists some element $w \neq 1$ such that $w \underset{\Gamma_W}{\sim} 1$. So there exists a sequence of elements $x_o = 1$, $x_1, \ldots, x_r = w$ with $r > 1$ such that for each j, $1 < j < r$, either $x_{j-1} \prec x_j$ or $x_j \prec x_{j-1}$, and either $\mathcal{L}(x_{j-1}) \not\subset \mathcal{L}(x_j)$ or $\mathcal{R}(x_{j-1}) \not\subset \mathcal{R}(x_j)$. But this is impossible since $\mathcal{L}(x_o) = \mathcal{R}(x_o) = \emptyset$ and then both $\mathcal{L}(x_o) \subset \mathcal{L}(x_1)$ and $\mathcal{R}(x_o) \subset \mathcal{R}(x_1)$. □

By Proposition 4.1, we see that $\{1\}$ is a 2-sided cell of $\tilde{A}_n$.

<u>Lemma 4.2</u> <u>Let $w \in C$. Then $|\mathcal{L}(w)| = |\mathcal{R}(w)| = 1$.</u>

<u>Proof</u>: By Lemma 2.2.5, we have $|\mathcal{L}(w)|$, $|\mathcal{R}(w)| > 1$ since $w \neq 1$. When $n = 2$, then also by Lemma 2.2.5, we have $|\mathcal{L}(w)|$, $|\mathcal{R}(w)| < 1$. So our result is true in this case. Now assume that $n > 3$. It is clear when $\ell(w) < 2$. So we may assume that $\ell(w) > 3$. If our assertion fails, then we can take a shortest element $w \in C$ such

that either $|\mathcal{L}(w)|$ or $|\mathcal{R}(w)|$ is greater than 1. By symmetry, we may assume that $|\mathcal{L}(w)| \geq 2$ and $w = s_r s_{r+1} \cdots s_t$ with $r < t-1$. Then there exists u with $\bar{u} \neq \bar{r}$ such that $\ell(s_u w) < \ell(w)$. By the exchange condition, there exists m with $r \leq m < t$ such that $s_u s_r s_{r+1} \cdots s_m = s_r s_{r+1} \cdots s_{m+1}$. So $\ell(s_u s_r s_{r+1} \cdots s_{m+1}) < \ell(s_r s_{r+1} \cdots s_{m+1})$. By assumption on $\ell(w)$, we must have $w = s_r s_{r+1} \cdots s_{m+1}$. We claim $\bar{u} = \overline{r \pm 1}$. Otherwise, $s_r s_u s_{r+1} \cdots s_m = s_r s_{r+1} \cdots s_{m+1}$ and it follows that $s_u s_{r+1} \cdots s_m = s_{r+1} \cdots s_{m+1}$. Hence $w' = s_{r+1} s_{r+2} \cdots s_{m+1} \in C$ with $\ell(w') < \ell(w)$ and $|\mathcal{L}(w')| \geq 2$. This contradicts our hypothesis. If $\bar{u} = \overline{r-1}$, then $s_{r-1} s_r \cdots s_m = s_r s_{r+1} \cdots s_{m+1}$. By symmetry, we have $s_r s_{r+1} \cdots s_{m+1} = s_{r-v} s_{r-v+1} \cdots s_{m+1-v}$ for any v, $1 \leq v < n$. Thus $\mathcal{L}(w) = \Delta$. This is impossible by Lemma 2.2.5. If $\bar{u} = \overline{r+1}$, then $s_{r+1} s_r s_{r+1} s_{r+2} \cdots s_m = s_r s_{r+1} \cdots s_{m+1}$. i.e. $s_r s_{r+1} s_r s_{r+2} \cdots s_m = s_r s_{r+1} \cdots s_{m+1}$. It follows that $s_r s_{r+2} s_{r+3} \cdots s_m = s_{r+2} \cdots s_{m+1}$. So $w'' = s_{r+2} s_{r+3} \cdots s_{m+1} \in C$ satisfies $\ell(w'') < \ell(w)$ and $|\mathcal{L}(w'')| \geq 2$. This also contradicts our hypothesis. So our result follows. □

<u>Lemma 4.3</u> <u>(i) C_t lies in some left cell of $\hat{A}_n$ for any t, $1 \leq t \leq n$.</u>

<u>(ii) C lies in some 2-sided cell of $\hat{A}_n$</u>

<u>Proof</u>: Let $w = s_r s_{r+1} \cdots s_t \in C_t$ for some r, $r < t$. Then by Lemma 4.2, $\mathcal{L}(w) = \{s_r\}$, $\mathcal{L}(s_{r+1} s_{r+2} \cdots s_t) = \{s_{r+1}\}$. So we have $w \underset{L}{\sim} s_{r+1} s_{r+2} \cdots s_t$. Similarly, we have $s_{r+1} s_{r+2} \cdots s_t \underset{L}{\sim} s_{r+2} \cdots s_t \underset{L}{\sim} \cdots \underset{L}{\sim} s_t$. Then $w \underset{L}{\sim} s_t$. By symmetry,

we can show that $s_{r'}s_{r'-1}\ldots s_t \underset{L}{\sim} s_t$ for any r', $r' > t$. Therefore, any element of C_t lies in the same left cell as s_t. (i) is proved.

Also, we can easily check that $s_t \underset{R}{\sim} s_t s_{t+1} \underset{L}{\sim} s_{t+1}$ for any t, $1 \leq t < n$. So Δ lies in some 2-sided cell of A_n. By the proof of (i), any element of C lies in the same left cell as some $s \in \Delta$. This implies that C is contained in some 2-sided cell of A_n. □

To show that C is a 2-sided cell of A_n, we need the following result which appears in [1, (2.3e), (2.3f)].

<u>Lemma 4.4</u> <u>Assume that W is a Coxeter group, $x,y \in W$ and s is a Coxeter generator of W.</u>

(i) <u>If $x < y$, $sy < y$, $sx > x$, then $x \prec y$ if and only if $y = sx$.</u>

(ii) <u>If $x < y$, $ys < y$, $xs > x$, then $x \prec y$ if and only if $y = xs$.</u> □

<u>Proposition 4.5</u> <u>(i) C is a 2-sided cell of A_n.</u>

(ii) <u>C_t is a left cell of A_n for any t, $1 \leq t \leq n$.</u>

(iii) <u>C is a union of n left (resp. right) cells of A_n.</u>

<u>Proof:</u> (i) It is enough to show that if $x \in C$, $y \notin C$, then $x \underset{\Gamma_W}{\not\sim} y$. By Proposition 4.1, we may assume that $y \neq 1$. If $x \underset{\Gamma_W}{\sim} y$, then there exists at least one pair x', y' with $x' \in C$, $y' \notin C$ satisfying either $x' \prec y'$ or $x' \succ y'$, and either $\mathcal{L}(x') \not\subseteq \mathcal{L}(y')$ or $\mathcal{R}(x') \not\subseteq \mathcal{R}(y')$. By symmetry, we may assume that $\mathcal{L}(x') \not\subseteq \mathcal{L}(y')$. Then Lemma 4.2 tells us that $\mathcal{L}(x') \cap \mathcal{L}(y') = \emptyset$. By Lemma 4.4(i), we have $x' = sy'$ for some $s \in \Delta$. We claim that $\ell(y') > \ell(x')$

since otherwise we would have $y' \in C$ and this contradicts our hypothesis. Without loss of generality, we may assume that $x' = s_r s_{r+1} \ldots s_t$ for some $r < t$. We claim $s \neq s_{r\pm1}$. For otherwise, y' must have one of the following forms:

(1) $s_{r-1} s_r s_{r+1} \ldots s_t$ (2) $s_{r+1} s_r$

(3) $s_{r+1} s_r s_{r+1} \ldots s_t$ with $r < t$.

But in cases (1), (2), we have $y' \in C$; in case (3), when $n = 2$, $y' \in C$, when $n > 2$, $\mathcal{L}(y') \supseteq \{s_r\} = \mathcal{L}(x')$. All these cases contradict our hypothesis. Then $s \neq s_{r\pm1}$. This implies that $y' = s_r s s_{r+1} \ldots s_t$ and hence $\{s_r\} \subseteq \mathcal{L}(y')$. i.e. $\mathcal{L}(x') \subseteq \mathcal{L}(y')$. This still contradicts our assumption. So we have proved (i).

(ii) For any t, t' with $\bar{t} \neq \bar{t}'$ and for any $y \in C_t$, $w \in C_{t'}$, we have $\mathcal{R}(y) = \{s_{\bar{t}}\} \neq \{s_{\bar{t}'}\} = \mathcal{R}(w)$. By Theorem A(i), this implies that $y \underset{L}{\not\sim} w$ and hence by Lemma 4.3, C_t and $C_{t'}$ belong to the different left cells of $\tilde{A}_n$. But by (i), $C = \bigcup_{t=1}^{n} C_t$ is a 2-sided cell of $\tilde{A}_n$ which must be a union of some left cells of $\tilde{A}_n$. It turns out that C_t is a left cell of $\tilde{A}_n$ for any t, $1 \leq t \leq n$.

(iii) By (i), (ii), we see that C is a union of n left cells of $\tilde{A}_n$. Since the map $w \to w^{-1}$ in $\tilde{A}_n$ induces a bijection between the set of left cells of $\tilde{A}_n$ and the set of right cells of $\tilde{A}_n$, and since C is invariant under this map, this implies that C is also a union of n right cells of $\tilde{A}_n$. □

Now we shall apply the above result to the affine Weyl group $\tilde{A}_2$.

<u>Proposition 4.6 In the affine Weyl group $\tilde{A}_2$, there are two 2-sided cells: $\{1\}$ and C, where C consists of non-identity elements of $\tilde{A}_2$. There are two left (resp. right) cells in C.</u>

<u>Proof</u>: Note that $\tilde{A}_2$ is actually an infinite dihedral group and that every non-identity element has one of the following forms:

(i) $s_1(s_2s_1)^{\ell}$ (ii) $s_2(s_1s_2)^{\ell}$

(iii) $(s_2s_1)^m$ (iv) $(s_1s_2)^m$

where $\ell \geq 0$, $m > 0$. Our result follows easily from Proposition 4.5. □

<u>Proposition 4.7 Let $\lambda = \{2 \geq 1 \geq \ldots \geq 1\}$, $\mu = \{1 \geq \ldots \geq 1\} \in \Lambda_n$. Then $\{1\} = \sigma^{-1}(\mu)$ and $C = \sigma^{-1}(\lambda)$.</u>

<u>Proof</u>: First we shall show that $\{1\} = \sigma^{-1}(\mu)$. It is clear that $\{1\} \subset \sigma^{-1}(\mu)$. On the other hand, if $w \neq 1$, let $w = s_t y$ with $s_t \in \Delta$ and $\ell(w) = \ell(s_t) + \ell(y)$. By Lemma 3.4, we have $\sigma(w) \geq \sigma(s_t)$. But it is easily seen that $\sigma(s_t) = \{2 \geq 1 \geq \ldots \geq 1\}$. So $w \notin \sigma^{-1}(\mu)$. Therefore $\{1\} = \sigma^{-1}(\mu)$.

Secondly, we shall show that $C = \sigma^{-1}(\lambda)$. When $n = 2$, we have $\Lambda_2 = \{\lambda, \mu\}$. It follows from $\{1\} = \sigma^{-1}(\mu)$ and $\tilde{A}_2 = \{1\} \cup C$ that $C = \sigma^{-1}(\lambda)$. Now assume $n \geq 3$. Then by the proof of Lemma 4.3, we see that C is actually a P-equivalence class of $\tilde{A}_n$. So by Proposition 3.9, this implies that $C \subset \sigma^{-1}(\lambda')$ for some $\lambda' \in \Lambda_n$. But we have $s_1 \in C$ and $\sigma(s_1) = \lambda$. Thus $\lambda' = \lambda$ and then $C \subset \sigma^{-1}(\lambda)$. Suppose $w \notin C$. Then there exists a reduced form $w = s_{i_1}s_{i_2}\ldots s_{i_t}$ with $t \geq 2$ such that w satisfies one of the following conditions:

(i) $\exists j$, $1 < j < t$, such that $\bar{i}_j \neq \overline{i_{j+1} \pm 1}$

(ii) $t > 3$ and $\exists j$, $1 < j < t$, such that $\overline{i_{j-1}} = \overline{i_{j+1}}$ and $\bar{i}_j = \overline{i_{j-1} \pm 1}$

In case (i), we have $\sigma(w) > \sigma(s_{i_j} s_{i_{j+1}}) = \{2 > 2 > 1 > \ldots > 1\} > \lambda$.

In case (ii), we have $\sigma(w) > \sigma(s_{i_{j-1}} s_{i_j} s_{i_{j+1}}) = \{3 > 1 > \ldots > 1\} > \lambda$.

So in both cases, $w \notin \sigma^{-1}(\lambda)$. This implies that $C = \sigma^{-1}(\lambda)$. □

<u>Remark 4.8</u> We can show that <u>when $w \in C$, the length function $\ell(w)$ has the much simplier form:</u>

$$\underline{\ell(w) = (t)w - (t+1)w - \left[\frac{(t)w - (t+1)w}{n}\right]}$$

<u>if $s_t \in \mathcal{L}(w)$.</u>

CHAPTER 5 : ITERATED STAR OPERATIONS AND INTERCHANGING OPERATIONS ON BLOCKS

In the remainder of our thesis, we shall always assume $n > 3$.

We wish to determine the left cell of A_n containing a given element w. We know from Theorem D that each P_L-equivalence class of A_n lies in some left cell of A_n. But any two elements of a P_L-equivalence class of A_n can be transformed from one to another by a succession of left star operations. It will turn out that we can perform various interchanging operations on blocks of w which are successions of left star operations and so give us elements in the same left cell as w. Although we cannot in general obtain all elements in the same left cell as w in this way, the interchanging operation on blocks will be crucial in our subsequent determination of the left cells.

§5.1 ITERATED STAR OPERATIONS

Assume that $w \in A_n$ has a DC form (A) at $i \in \mathbb{Z}$ with $|A(w)| = r$, $1 < r < n$. Assume that the entry $e(i,j(w))$ of w satisfies $j(w) < j_A^1(w)$, where $e((w), j_A^h(w))$ is the h-th entry of $A(w)$ for $1 < h < r$. Let $k = \max\{h \mid 1 < h < r,\ j_A^h(w) > j(w)\}$. Then there exists a sequence of elements $x_o = w, x_1, \ldots, x_{r-1}$, in A_n such that for every $1 < t < r-1$, we have $x_t = {}^*x_{t-1}$ in $D_L(s_{i+t-1})$. In particular, $w' = x_{r-1}$ has a DC form (A) at $i-1$ with $|A(w')| = r$ such that

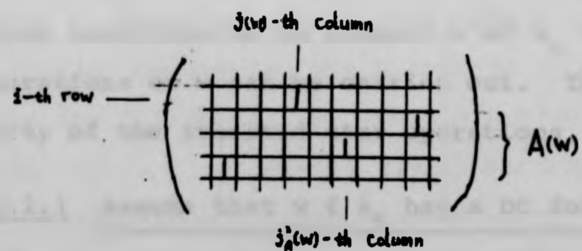
$$j_A^u(w') = \begin{cases} j_A^u(w) & \text{for } 1 < u < r,\ u \neq k \\ j(w) & \text{for } u = k. \end{cases}$$

where $e((w'),\ j_A^h(w'))$ is the h-th entry of $A(w')$ for $1 < h < r$. and the entry $e(i+r,\ j(w'))$ of w' satisfies $j(w') = j_A^k(w)$.

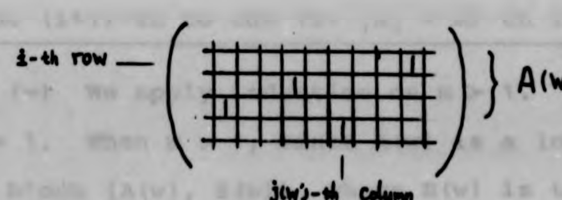
Definition 5.1.1 For $w, w' \in A_n$, $1 < r < n$ and $i \in \mathbb{Z}$, we write $w' \xleftarrow{{}^*(i+1,r)} w$, if (i) w has a DC form (A) at i with $|A(w)| = r$. (ii) $\exists$ a sequence of elements $x_o = w, x_1, \ldots, x_{r-1} = w'$ in A_n such that for every $1 < h < r-1$, we have $x_h = {}^*x_{h-1}$ in $D_L(s_{i+h-1})$.

For $w, w' \in A_n$, $i \in \mathbb{Z}$ and $1 < r < r+m < n$, we write $w' \xleftarrow{{}^*(i+1,r,m)} w$, if $\exists$ a sequence of elements $x_o = w$, $x_1, \ldots, x_m = w'$ in A_n such that for every $1 < h < m$, we have $x_h \xleftarrow{{}^*(i+2-h,r)} x_{h-1}$. We write $w \xrightarrow{{}^*(i+1,r,m)} w'$ if $w \xleftarrow{{}^*(i+1-m,r,m)} w'$.

Here is an example for $w \in A_n$ which has DC form (A) at $i \in \mathbb{Z}$ with $|A(w)| = 3$.



Then there exists w' with $w' \xleftarrow{{}^*(i+1,3)} w$ as follows:



where $j(w') = j_A^2(w)$.

Remark 5.1.2 (i) If w has a DC form (A) at $i \in \mathbb{Z}$ with $|A(w)| = r < n$, then we can easily check that w' with $w' \xleftarrow{*(i+1,r)} w$ exists if and only if $j_o < j_A^1(w)$. Also, w" with $w \xrightarrow{*(i+1,r)} w''$ exists if and only if $j_A^r(w) < j_{r+1}$, where $e(i,j_o)$, $e(i+r+1,j_{r+1})$ are the entries of w.

(ii) Given $w \in A_n$, $i \in \mathbb{Z}$, $r,m > 0$ with $r + m < n$, an element w' $w' \xleftarrow{*(i+1,r,m)} w$ or $w \xrightarrow{*(i+1,r,m)} w'$ is unique if it exists.

(iii) In general, the expression $w \xrightarrow{*(i+1,r,m)} w'$ is equivalent to $w \xleftarrow{*(i+1-m,r,m)} w'$ but not to $w' \xleftarrow{*(i+1,r,m)} w$.

§5.2 SOME RESULTS ON ITERATED STAR OPERATIONS

In this section, the first two lemmas give a necessary and sufficient condition on an element w of A_n for which the iterated star operations on w can be carried out. The last lemma states a property of the iterated star operations.

Lemma 5.2.1 Assume that $w \in A_n$ has a DC form (A) at $i \in \mathbb{Z}$. Let m satisfy $1 \leq m \leq n-|A|$. Then there exists w' with $w \xrightarrow{*(i+1,|A|,m)} w'$ if and only if A(w) is a longest descending chain in A'(w), where A'(x) is the block of x consisting of rows from the (i+1)-th to the (i+ |A| + m)-th for $x \in A_n$.

Proof: ($\Leftarrow$) We apply induction on $m \geq 1$. The result is obvious for m = 1. When m > 1, since A(w) is a longest descending chain in the block [A(w), B(w)], where B(w) is the (i + |A| + 1)-th row of w, there exists w_1 such that $w \xrightarrow{*(i+1,|A|)} w_1$. By the proof of Lemma 3.7, $A(w_1)$ is a longest descending chain in $A'(w_1)$ and so also in $A''(w_1)$, where $A(w_1)$ (resp. $A''(w_1)$) is the block of w_1 consisting of rows from the $(i+2)$-th to the $(i+1+|A|)$-th (resp. from the $(i+2)$-th to the $(i + |A| + m)$-th). By inductive hypothesis, there exists w' such that $w_1 \xrightarrow{*(i+2,|A|,m-1)} w'$. So we have $w \xrightarrow{*(i+1,|A|,m)} w'$.

($\Rightarrow$) We also apply induction on $m \geq 1$. It is true for $m = 1$. Now assume $m > 1$. Since w' exists, it implies that there exists w_1 such that $w \xrightarrow{*(i+1,|A|)} w_1$ and $w_1 \xrightarrow{*(i+2,|A|,m-1)} w'$. By inductive hypothesis, $A(w_1)$ is a longest descending chain in the block $[A(w_1), C(w_1)]$, where $C(w_1)$ is the block of w_1 consisting of rows from the $(i + |A| + 2)$-th to the $(i + |A| + m)$-th. By Lemma 3.7, to reach our goal, it suffices to show that $A(w_1)$ is also a longest descending chain in $A'(w_1)$, or equivalently to show that for any descending chain $D \subseteq A'(w_1)$, there exists a descending chain $D' \subseteq [A(w_1), C(w_1)]$ such that $|D'| = |D|$. Now suppose we are given a fixed descending chain $D \subseteq A'(w_1)$. If $D \subseteq [A(w_1), C(w_1)]$, then let $D' = D$. If $D \not\subseteq [A(w_1), C(w_1)]$, let $D = \{e_{w_1}(i_t,j_t) | 1 \leq t \leq \ell, i_1 < \dots < i_\ell, j_1 > \dots > j_\ell\}$. Then $i_1 = i + 1$. Let $e_{w_1}(i_o,j_o)$ be the first entry of $A(w_1)$. Since $i_o = i_1+1 > i_1$ and $j_o > j_1$, this implies that $e_{w_1}(i_o,j_o) \notin D$. But by $i_1 < i_2$, $i_o = i_1+1$ and $i_2 \neq i_1+1$, we have $i_o < i_2$. Clearly, $j_o > j_1 > j_2$. Let $D' = (\{e_{w_1}(i_o,j_o)\} \cup D) - \{e_{w_1}(i_1,j_1)\}$. Then $D' \subseteq [A(w_1), C(w_1)]$ is as required. $\square$

Lemma 5.2.2 Assume that $w \in \mathbb{A}_n$ has a DC form (A) at $i \in \mathbb{Z}$. Suppose m satisfies $1 < m < n-|A|$. Then there exists w' with $w' \xleftarrow{*(i+1,|A|,m)} w$ if and only if A(w) is a longest descending chain in A'(w) where A'(x) is the block of x consisting of rows from the (i+1-m)-th to the (i+|A|)-th for $x \in \mathbb{A}_n$.

Proof: ($\Rightarrow$) We know that the existence of $w' \in \mathbb{A}_n$ with $w' \xleftarrow{*(i+1,|A|,m)} w$ is equivalent to the existence of $w' \in \mathbb{A}_n$ with $w' \xrightarrow{*(i+1-m,|A|,m)} w$ and that when w' does exist, w' has a DC form (A) at i-m with $|A(w')| = |A(w)|$. Now we start with w'. Then since w with $w' \xrightarrow{*(i+1-m,|A|,m)} w$ exists, we see by Lemma 5.2.1 that A(w') is a longest descending chain in A'(w'). So A(w) is a longest descending chain in A'(w) by the proof of Lemma 3.7.

($\Leftarrow$) Imitate the proof of the corresponding part of Lemma 5.2.1. □

Lemma 5.2.3 Assume that $w \in \mathbb{A}_n$ has a DC form (A) at $i \in \mathbb{Z}$ with $|A(w)| = r$, $1 < r < n-2$. Let $e(i-1,j_1(w))$, $e(i,j_2(w))$ be entries of w. If $w' \in \mathbb{A}_n$ with $w' \xleftarrow{*(i+1,r,2)} w$ exists, let $e(i+r-1, j_1(w'))$, $e(i+r, j_2(w'))$ be entries of w'. Then

(i) $j_1(w) > j_2(w) \Longleftrightarrow j_1(w') > j_2(w')$

(ii) $j_1(w) < j_2(w) \Longleftrightarrow j_1(w') < j_2(w')$.

Proof: It is enough to show the implication in the direction "$\Rightarrow$" for both cases.

(i) Let $a_2 = \max\{h \mid 1 < h < r,\ j_A^h(w) > j_2(w)\}$

$$a_1 = \max\{h \mid 1 < h < a_2,\ j_A^h(w) > j_1(w)\}$$

Then since w' exists, this implies that a_1, a_2 both exists with $j_1(w') = j_A^{a_1}(w)$ and $j_2(w') = j_A^{a_2}(w)$. Since $a_1 < a_2$, we have $j_1(w') > j_2(w')$.

(ii) We may assume that $r > 2$, since otherwise, the result is obvious. Let $a = \max\{h \mid 1 < h < r,\ j_A^h(w) > j_2(w)\}$. Then $j_2(w') = j_A^a(w)$. Since $j_1(w') = \min\{j_2(w),\ j_A^h(w) \mid 1 < h < r,\ j_A^h(w) > j_1(w)\}$, it follows that $j_1(w') < j_2(w) > j_2(w')$. □

§5.3 THE INTERCHANGING OPERATIONS $\rho_{A_2}^{A_1}$ AND $\theta_{A_1}^{A_2}$.

In this section, we shall first define the interchanging operations $\rho_{A_2}^{A_1}$ and $\theta_{A_1}^{A_2}$ on an element $w \in A_n$ and then give a necessary and sufficient condition for which the interchanging operations on w can be carried out. We also give the formulae to calculate these operations. One special case when w has a local MDC form which is quasi-normal (resp. normal) for the first k layers, $k > 1$, is most interesting for us. In that case, ρ operations on w have a very good behaviour.

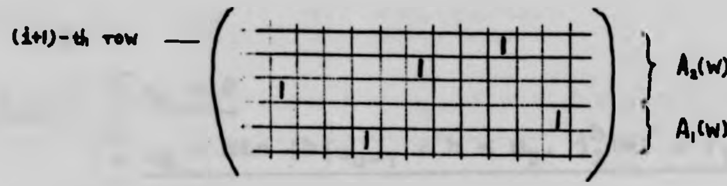
We assume in this section that $w \in A_n$ always has a local MDC form (A_2, A_1) at $i \in \mathbb{Z}$ with $|A_t| = m_t$, $t = 1,2$, unless the contrary is specified.

Definition 5.3.1 Set $w' = \rho_{A_2}^{A_1}(w)$ if $\exists$ w' satisfying $w \xrightarrow{*(i+1,m_2,m_1)} w'$ and $w'' = \theta_{A_1}^{A_2}(w)$ if $\exists$ w'' satisfying $w'' \xleftarrow{*(i+m_2+1,m_1,m_2)} w$.

Note that such elements w', w" do not always exist in general. But when w' (resp. w") does exist, then by Lemma 5.2.3, w' (resp. w") has a local MDC form (A_1', A_2') at i with $|A_t'| = m_t$, $t = 1,2$. We shall always assume that w' (resp. w") has a local MDC form (A_1, A_2) at i unless the contrary is specified. Clearly, in that case, $w = \theta_{A_2}^{A_1} \cdot \rho_{A_2}^{A_1}(w)$ (resp. $w = \rho_{A_1}^{A_2} \cdot \theta_{A_1}^{A_2}(w)$). For convenience we admit blocks occurring in $\rho_{A_2}^{A_1}$ or $\theta_{A_1}^{A_2}$ which have the size 0. We make a convention that $w = \rho_{A_2}^{A_1}(w)$ and $w = \theta_{A_1}^{A_2}(w)$ when either $|A_1| = 0$ or $|A_2| = 0$.

When w' does exist, we define a sequence $\xi(w, \rho_{A_2}^{A_1})$: $x_{o,m_2-1} = w$, $x_{11}, x_{12}, \ldots, x_{1,m_2-1}$, $x_{21}, x_{22}, \ldots, x_{2,m_2-1}, \ldots, x_{m_1,1}, x_{m_1,2}, \ldots, x_{m_1,m_2-1}$ such that $x_{hj} = {}^*x_{h,j-1}$ in $\mathcal{D}_L(s_{i+h-1+m_2-j})$ for h, j with $1 < h < m_1$ and $1 < j < m_2-1$, and $x_{h+1,1} = {}^*x_{h,m_2-1}$ in $\mathcal{D}_L(s_{i+h-1+m_2})$ for $0 < h < m_1$. Then $\xi(w, \rho_{A_2}^{A_1})$ exists with $w' = x_{m_1,m_2-1}$. We call $\xi(w, \rho_{A_2}^{A_1})$ the sequence corresponding to $\rho_{A_2}^{A_1}$ on w. Similarly, we can define the sequence $\xi(w, \theta_{A_1}^{A_2})$ corresponding to $\theta_{A_1}^{A_2}$ on w when w" exists.

For example, let w have a local MDC form (A_2, A_1) at $i \in \mathbb{Z}$ with $|A_2| = 3$ and $|A_1| = 2$ as follows:



Then $w' = \rho^{\lambda_1}_{\lambda_2}(w)$ has the following form

and the sequence corresponding to $\rho^{\lambda_1}_{\lambda_2}$ on w is $\xi(w,\rho^{\lambda_1}_{\lambda_2})$: $x_{0,2} = w$, x_{11}, x_{12}, x_{21}, $x_{22} = w'$, where $x_{11} = {}^*x_{02}$ in $D_L(s_{i+2})$, $x_{12} = {}^*x_{11}$ in $D_L(s_{i+1})$, $x_{21} = {}^*x_{12}$ in $D_L(s_{i+3})$ and $x_{22} = {}^*x_{21}$ in $D_L(s_{i+2})$.

In the above example, we also have $w = \theta^{\lambda_1}_{\lambda_2}(w')$ and the sequence corresponding to $\theta^{\lambda_1}_{\lambda_2}$ on w' is obtained from the sequence $\xi(w,\rho^{\lambda_1}_{\lambda_2})$ just by reversing its order.

Now we shall give the formulae for calculation of $\rho^{\lambda_1}_{\lambda_2}(w)$ and $\theta^{\lambda_2}_{\lambda_1}(w)$, and also give a necessary and sufficient condition on the existence of $\rho^{\lambda_1}_{\lambda_2}(w)$ or $\theta^{\lambda_2}_{\lambda_1}(w)$.

Let

$$(5.3.2)\quad \begin{cases} \alpha_0 = 0 \\ \alpha_u = \min\,\{h \mid \alpha_{u-1} < h \leq m_2,\ j_2^h(w) < j_1^u(w)\},\ 1 \leq u \leq m_1 \end{cases}$$

and

$$(5.3.3)\quad \begin{cases} \beta_{m_2+1} = m_1+1 \\ \beta_u = \max\,\{h \mid 1 \leq h < \beta_{u+1},\ j_1^h(w) > j_2^u(w)\},\ 1 \leq u \leq m_2 \end{cases}$$

<u>Lemma 5.3.4</u> <u>When $|A_2(w)| \neq 0$, we have</u>

$$w' = \rho_{A_2}^{A_1}(w) \text{ exists} \Longleftrightarrow \text{all } \alpha_u,\ 1 \leq u \leq m_1, \text{ exist.}$$

<u>When they do exist, we get</u>

$$\begin{cases} j_1^u(w') = j_2^{\alpha_u}(w) & \text{if } 1 \leq u \leq m_1 \\ j_2^v(w') = \begin{cases} j_2^v(w) & \text{if } 1 \leq v \leq m_2,\ v \notin \{\alpha_1,\ldots,\alpha_{m_1}\} \\ j_1^h(w) & \text{if } v = \alpha_h \text{ for some } 1 \leq h \leq m_1 \end{cases} \end{cases}$$

<u>Proof</u>: By definition of $\rho_{A_2}^{A_1}$. □

<u>Lemma 5.3.5</u> <u>When $|A_2(w)| \neq 0$, we have</u>

$$w' = \rho_{A_2}^{A_1}(w) \text{ exists} \Longleftrightarrow A_2(w) \text{ is a longest descending chain in the block } [A_2(w),\ A_1(w)].$$

<u>Proof</u>: By Lemma 5.2.1 and the definition of $\rho_{A_2}^{A_1}$. □

Lemma 5.3.6 When $|A_1(w)| \neq 0$, we have

$$w'' = \theta_{A_1}^{A_2}(w) \text{ exists} \iff \text{all } \beta_u,\ 1 < u < m_2, \text{ exist.}$$

When they do exist, we get

$$\begin{cases} j_2^u(w'') = j_1^{\beta_u}(w) & \text{if } 1 < u < m_2 \\ j_1^v(w'') = \begin{cases} j_1^v(w) & \text{if } 1 < v < m_1,\ v \notin \{\beta_1,\ldots,\beta_{m_2}\} \\ j_2^h(w) & \text{if } v = \beta_h \text{ for some } 1 < h < m_2 \end{cases} \end{cases}$$

Proof: By definition of $\theta_{A_1}^{A_2}$. □

Lemma 5.3.7 When $|A_1(w)| \neq 0$, we have

$$w'' = \theta_{A_1}^{A_2}(w) \text{ exists} \iff A_1(w) \text{ is a longest descending chain in the block } [A_2(w), A_1(w)].$$

Proof: By Lemma 5.2.2 and the definition of $\theta_{A_1}^{A_2}$. □

In the remainder of this section, we shall introduce the concept of being quasi-normal (resp. normal) for the first k layers of a local MDC form of w. Before doing this, we give some definitions.

Definition 5.3.8 For any $\ell, i \in \mathbb{Z}$ with $\ell > 0$ and $1 < i < \ell$, suppose that either $a_i \in \mathbb{Z}$ or a_i does not exist. We define $(a_1,\ldots,a_\ell)^{(om)} \in \bigcup_{t=0}^{\ell} \mathbb{Z}^t$ as follows:

(i) $(a_1,\ldots,\hat{a}_j,\ldots,a_\ell)^{(om)} = (a_1,\ldots,a_\ell)^{(om)}$, if a_j does not exist for some $1 \le j \le \ell$, where the notation $\hat{a}_j$ means that we omit the term a_j.

(ii) $(a_1,\ldots,a_\ell)^{(om)} = (a_1,\ldots,a_\ell)$, if all a's exist with the convention that $\mathbb{Z}^0 = \emptyset$.

<u>Definition 5.3.9</u> For any ℓ, $i \in \mathbb{Z}$ with $\ell \ge 0$ and $1 \le i \le \ell$, suppose that either $a_i \in \mathbb{Z}$ or a_i does not exist. We define $(a_1 < \ldots < a_\ell)^{(om)}$ as follows:

(i) Let $(a_1 < \ldots < a_\ell)^{(om)}$ be $(a_1 < \ldots < \hat{a}_j < \ldots < a_\ell)^{(om)}$, if a_j does not exist for some $1 \le j \le \ell$.

(ii) Let $(a_1 < \ldots < a_\ell)^{(om)}$ be $a_1 < \ldots < a_\ell$, if all a's exist with the convention that $(a_1 < \ldots < a_\ell)^{(om)}$ is an identity when $\ell \le 1$.

Suppose that w has a local MDC form $(A_\ell,\ldots,A_1)$ at $i \in \mathbb{Z}$. For any $u \ge 1$, the u-th layer with respect to this local MDC form is, by definition, the set of entries of w consisting of all the u-th entries of $A_t(w)$, $1 \le t \le \ell$, and their congruent entries.

<u>Definition 5.3.10</u> Suppose that w has a local MDC form $(A_\ell,\ldots,A_1)$ at $i \in \mathbb{Z}$. Let k satisfy $1 \le k \le \max\{|A_t| \mid 1 \le t \le \ell\}$. If for any h, $1 \le h \le k$, $(j_\ell^h(w) < j_{\ell-1}^h(w) < \ldots < j_1^h(w))^{(om)}$ holds, then we say that w has a local MDC form $(A_\ell,\ldots,A_1)$ at i which is quasi-normal for the first k layers. If for any h, $1 \le h \le k$, $(j_1^h(w) - n < j_\ell^h(w) < j_{\ell-1}^h(w) < \ldots < j_1^h(w))^{(om)}$ holds, then we say that w has a local MDC form $(A_\ell,\ldots,A_1)$ at i which is normal

for the first k layers. In particular, in the above definition, if $k = \max\{|A_t| \mid 1 \le t \le \ell\}$, we shall say that w has a local MDC form $(A_\ell,\ldots,A_1)$ at i which is quasi-normal (resp. normal).

Now assume that w has a local MDC form (A_2,A_1) at i which is quasi-normal for the first k layers with $k = \min\{m_1,m_2\}$. Then in Formula (5.3.2), we have $\alpha_v = v$ for all $1 \le v \le k$. Let $\alpha'_u = \alpha_{u+k}-k$ for any $0 \le u \le m_1-k$. Then we have

$$(5.3.11)\quad \begin{cases} \alpha'_0 = 0 \\ \alpha'_u = \min\{h \mid \alpha'_{u-1} < h \le m_2-k,\ j_2^{k+h}(w) < j_1^{k+u}(w)\},\quad 1 \le u \le m_1-k \end{cases}$$

Corollary 5.3.12 Assume that $w \in A_n$ has a local MDC form (A_2,A_1) at $i \in \mathbb{Z}$ with $|A_2| \ne 0$ which is quasi-normal for the first k layers for some $k \ge 0$. Then

$$w' = \rho_{A_2}^{A_1}(w) \text{ exists} \iff \text{all } \alpha_u,\ k+1 \le u \le m_1, \text{ exist.}$$

When they do exist, we have

$$\begin{cases} (j_1^h(w'),\ j_2^h(w')) = (j_2^h(w),\ j_1^h(w)), & \text{if } 1 \le h \le k \\ j_1^u(w') = j_2^{\alpha_u}(w), & \text{if } k+1 \le u \le m_1 \\ j_2^v(w') = \begin{cases} j_2^v(w), & \text{if } k+1 \le v \le m_2 \text{ and } v \notin \{\alpha_{k+1},\ldots,\alpha_{m_1}\} \\ j_1^h(w), & \text{if } v = \alpha_h \text{ for some } k+1 \le h \le m_1 \end{cases} \end{cases}$$

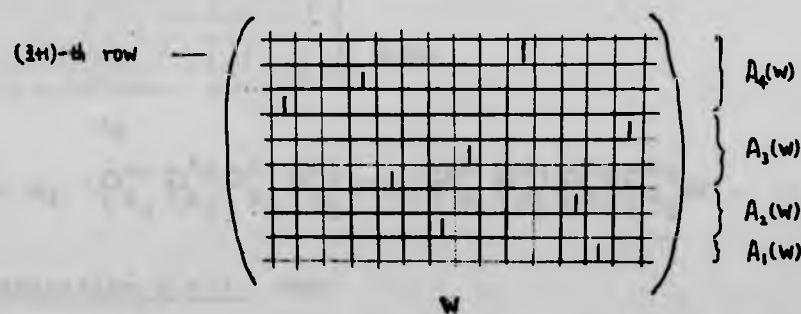
Proof: This follows from Lemma 5.3.4. □

We can regard Lemma 5.3.4 as a special case of Corollary 5.3.12 when $k = 0$.

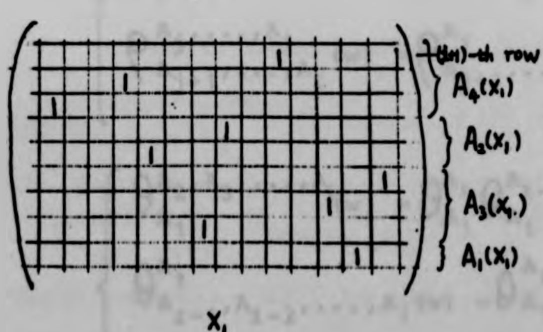
§5.4 MORE GENERAL INTERCHANGING OPERATIONS

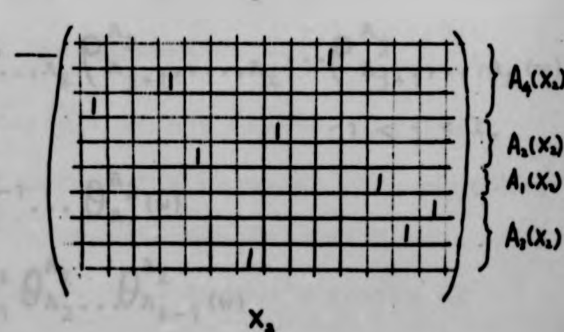
In this section, we assume that $w \in \mathbf{A}_n$ has a local MDC form $(A_\ell, \ldots, A_1)$ at $i \in \mathbb{Z}$. We shall extend the results of §5.3 to more general cases.

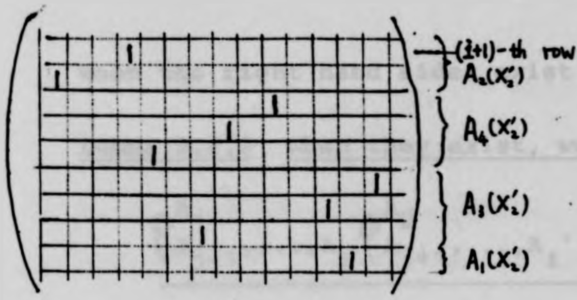
First let us observe an example: Assume that $w \in \mathbf{A}_n$ has a local MDC form (A_4, A_3, A_2, A_1) at $i \in \mathbb{Z}$.



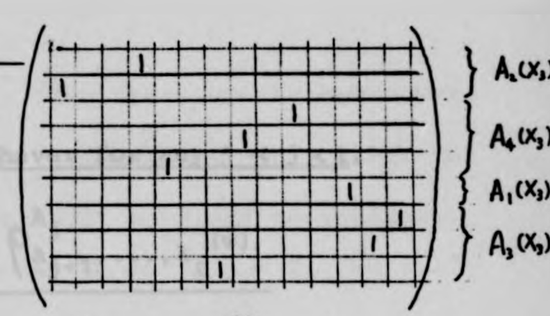
Then there exist elements x_1, x_2, x_2', x_3 and x_4 such that $x_1 = \rho^{A_2}_{A_3}(w)$, $x_2 = \rho^{A_1}_{A_3}(x_1)$, $x_2' = \rho^{A_2}_{A_4}(x_1)$, $x_3 = \rho^{A_2}_{A_4}(x_2) = \rho^{A_1}_{A_3}(x_2')$ and $x_4 = \rho^{A_1}_{A_4}(x_3)$ are as follows:



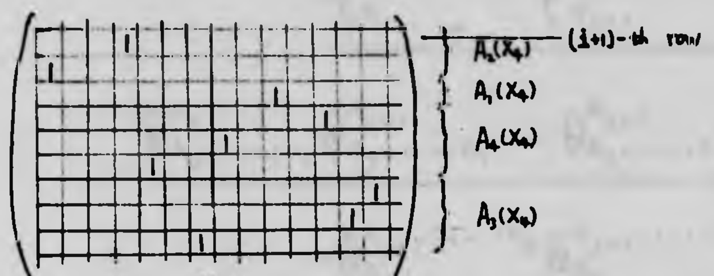




X'_2



X_3



X_4

So $x_4 = \rho^{\lambda_1}_{\lambda_4}\rho^{\lambda_2}_{\lambda_4}\rho^{\lambda_1}_{\lambda_3}\rho^{\lambda_2}_{\lambda_3}(w) = \rho^{\lambda_1}_{\lambda_4}\rho^{\lambda_1}_{\lambda_3}\rho^{\lambda_2}_{\lambda_4}\rho^{\lambda_2}_{\lambda_3}(w)$.

__Definition 5.4.1__ Set

$$\begin{cases} \rho^{\lambda_1}_{\lambda_2,\lambda_3,\dots,\lambda_\ell}(w) = \rho^{\lambda_1}_{\lambda_\ell}\rho^{\lambda_1}_{\lambda_{\ell-1}}\cdots\rho^{\lambda_1}_{\lambda_2}(w) \\ \rho^{\lambda_{\ell-1},\lambda_{\ell-2},\dots,\lambda_1}_{\lambda_\ell}(w) = \rho^{\lambda_1}_{\lambda_\ell}\rho^{\lambda_2}_{\lambda_\ell}\cdots\rho^{\lambda_{\ell-1}}_{\lambda_\ell}(w) \\ \rho^{\lambda_j,\dots,\lambda_1}_{\lambda_{j+1},\dots,\lambda_\ell}(w) = \rho^{\lambda_1}_{\lambda_{j+1},\dots,\lambda_\ell}\rho^{\lambda_2}_{\lambda_{j+1},\dots,\lambda_\ell}\cdots\rho^{\lambda_j}_{\lambda_{j+1},\dots,\lambda_\ell}(w), \\ \qquad 1 < j < \ell. \end{cases}$$

$$\begin{cases} \theta^{\lambda_2,\lambda_3,\dots,\lambda_\ell}_{\lambda_1}(w) = \theta^{\lambda_\ell}_{\lambda_1}\theta^{\lambda_{\ell-1}}_{\lambda_1}\cdots\theta^{\lambda_2}_{\lambda_1}(w) \\ \theta^{\lambda_\ell}_{\lambda_{\ell-1},\lambda_{\ell-2},\dots,\lambda_1}(w) = \theta^{\lambda_\ell}_{\lambda_1}\theta^{\lambda_\ell}_{\lambda_2}\cdots\theta^{\lambda_\ell}_{\lambda_{\ell-1}}(w) \\ \theta^{\lambda_{j+1},\dots,\lambda_\ell}_{\lambda_j,\dots,\lambda_1}(w) = \theta^{\lambda_\ell}_{\lambda_j,\dots,\lambda_1}\theta^{\lambda_{\ell-1}}_{\lambda_j,\dots,\lambda_1}\cdots\theta^{\lambda_{j+1}}_{\lambda_j,\dots,\lambda_1}(w), \\ \qquad 1 < j < \ell. \end{cases}$$

when the right hand sides exist.

<u>Lemma 5.4.2</u> <u>When they exist, we have: for any $1 \le j \le \ell$,</u>

$$\frac{\rho^{A_1}_{A_{j+1},\dots,A_\ell}\,\rho^{A_2}_{A_{j+1},\dots,A_\ell}\cdots\rho^{A_j}_{A_{j+1},\dots,A_\ell}(w)}{\qquad} = \frac{\rho^{A_j,\dots,A_1}_{A_\ell}\;\rho^{A_j,\dots,A_1}_{A_{\ell-1}}\cdots\rho^{A_j,\dots,A_1}_{A_{j+1}}(w)}{\qquad}$$

$$\frac{\theta^{A_\ell}_{A_j,\dots,A_1}\,\theta^{A_{\ell-1}}_{A_j,\dots,A_1}\cdots\theta^{A_{j+1}}_{A_j,\dots,A_1}(w)}{\qquad} = \frac{\theta^{A_{j+1},\dots,A_\ell}_{A_1}\,\theta^{A_{j+1},\dots,A_\ell}_{A_2}\cdots\theta^{A_{j+1},\dots,A_\ell}_{A_j}(w)}{\qquad}$$

<u>Proof</u>: By definition of ρ,θ it suffices to show that when they exist, $\rho^{A_{j_1}}_{A_{i_1}}\rho^{A_{j_2}}_{A_{i_2}}(w) = \rho^{A_{j_2}}_{A_{i_2}}\rho^{A_{j_1}}_{A_{i_1}}(w)$ and $\theta^{A_{i_1}}_{A_{j_1}}\theta^{A_{i_2}}_{A_{j_2}}(w) = \theta^{A_{i_2}}_{A_{j_2}}\theta^{A_{i_1}}_{A_{j_1}}(w)$ hold, where $j_1, j_2, i_1, i_2 \in \{1,2,\dots,\ell\}$ are distinct. But this is obvious. □

As in the above example, we can rewrite x_4 by $\rho^{A_2,A_1}_{A_4}\rho^{A_2,A_1}_{A_3}(w)$ or by $\rho^{A_1}_{A_3,A_4}\rho^{A_2}_{A_3,A_4}(w)$.

The above lemma tells us that the expression of $\rho^{A_j,\dots,A_1}_{A_{j+1},\dots,A_\ell}$ (or $\theta^{A_{j+1},\dots,A_\ell}_{A_j,\dots,A_1}$) in terms of ρ^A_B (or θ^A_B) is not unique. So for the sake of definiteness, we shall define the sequence corresponding to $\rho^{A_j,\dots,A_1}_{A_{j+1},\dots,A_\ell}$ (resp. $\theta^{A_{j+1},\dots,A_\ell}_{A_j,\dots,A_1}$) on w in the following way. These sequences will be used in Chapter 9 to define a family of much longer and also more important sequences.

If $w' = \rho^{A_1}_{A_2,\ldots,A_\ell}(w)$ exists, then there exists sequences corresponding to $\rho^{A_1}_{A_j}$ on $\rho^{A_1}_{A_2,\ldots,A_{j-1}}(w)$ for $2 < j < \ell$ (For the definition of these sequences, see §5.3) with the convention that $\rho^{A_1}_{A_2,\ldots,A_{j-1}}$ is the identity map on w for j = 2. We call $\xi(w,\rho^{A_1}_{A_2,\ldots,A_\ell})$ the sequence corresponding to $\rho^{A_1}_{A_2,\ldots,A_\ell}$ on w if it is obtained by linking all the above sequences together. Furthermore, we call $\xi(w,\rho^{A_j,\ldots,A_1}_{A_{j+1},\ldots,A_\ell})$ the sequence corresponding to $\rho^{A_j,\ldots,A_1}_{A_{j+1},\ldots,A_\ell}$ on w if it is obtained by linking all the sequences $\xi(\rho^{A_j,\ldots,A_t}_{A_{j+1},\ldots,A_\ell}(w), \rho^{A_{t-1}}_{A_{j+1},\ldots,A_\ell})$, $1 < t < j+1$, together, with the convention that $\rho^{A_j,\ldots,A_t}_{A_{j+1},\ldots,A_\ell}(w) = w$ for $t = j+1$. We can define $\xi(w, \theta^{A_{j+1},\ldots,A_\ell}_{A_j,\ldots,A_1})$ as the sequence corresponding to $\theta^{A_{j+1},\ldots,A_\ell}_{A_j,\ldots,A_1}$ on w to be obtained from the sequence $\xi(\theta^{A_{j+1},\ldots,A_\ell}_{A_j,\ldots,A_1}(w), \rho^{A_1,\ldots,A_j}_{A_\ell,\ldots,A_{j+1}})$ by reversing its order.

The following two lemmas generalize Lemmas 5.3.5 and 5.3.7.

Lemma 5.4.3 Assume that $w \in A_n$ has a local MDC form $(A_\ell,\ldots,A_1)$ at $i \in \mathbb{Z}$. Then $w' = \rho^{A_{\ell-1},A_{\ell-2},\ldots,A_1}_{A_\ell}(w)$ exists if and only if $A_\ell(w)$ is a longest descending chain in the block $[A_\ell(w),\ldots,A_1(w)]$.

Proof: Since $w \xrightarrow{*(i+4),|A_\ell|,\ \sum_{t=1}^{\ell-1}|A_t|)} w'$, this follows from Lemma 5.2.1. □

<u>Lemma 5.4.4</u> <u>Assume that $w \in A_n$ has a local MDC form $(A_\ell,\ldots,A_1)$ at $i \in \mathbb{Z}$. Then $w' = \rho_{A_1}^{A_2,A_3,\ldots,A_\ell}(w)$ exists if and only if $A_1(w)$ is a longest descending chain in the block $[A_\ell(w),\ldots,A_1(w)]$.</u>

<u>Proof</u>: ($\Rightarrow$) The existence of w' implies that $w = \rho_{A_1}^{A_\ell,A_{\ell-1},\ldots,A_2}\ \theta_{A_1}^{A_2,A_3,\ldots,A_\ell}(w) = \rho_{A_1}^{A_\ell,A_{\ell-1},\ldots,A_2}(w')$. By Lemma 5.4.3, it follows that $A_1(w')$ is a longest descending chain in the block $[A_\ell(w'),\ldots,A_1(w')]$ and then $A_1(w)$ is a longest descending chain in the block $[A_\ell(w),\ldots,A_1(w)]$ by Lemma 3.7.

($\Leftarrow$) This is proved as in the proof of Lemma 5.2.1. □

The next lemma and its corollary show that ρ operations preserve the property of being quasi-normal (resp. normal) for the first k layers of a local MDC form.

<u>Lemma 5.4.5</u> <u>Assume that $w \in A_n$ has a local MDC form $(A_{1\ell_1},\ldots,A_{11},A_{2\ell_2},\ldots,A_{21},A_{3\ell_3},\ldots,A_{31},A_{4\ell_4},\ldots,A_{41})$ at $i \in \mathbb{Z}$ which is quasi-normal (resp. normal) for the first k layers. If there exists $w' = \rho_{A_{21},\ldots,A_{2\ell_2}}^{A_{3\ell_3},\ldots,A_{31}}(w)$, then for any h with $1 < h < k$, we have $(j_{1\ell_1}^h(w'),\ldots,j_{11}^h(w'),\ j_{3\ell_3}^h(w'),\ldots,j_{31}^h(w'),j_{2\ell_2}^h(w'),\ldots,j_{21}^h(w'),j_{4\ell_4}^h(w'),\ldots,j_{41}^h(w'))^{(om)} = (j_{1\ell_1}^h(w),\ldots,j_{11}^h(w),\ j_{2\ell_2}^h(w),\ldots,j_{21}^h(w),j_{3\ell_3}^h(w),\ldots,j_{31}^h(w),j_{4\ell_4}^h(w),\ldots,j_{41}^h(w))^{(cm)}$. In particular, w' has a local MDC form $(A_{1\ell_1},\ldots,A_{11},\ A_{3\ell_3},\ldots,A_{31},A_{2\ell_2},\ldots,A_{21},A_{4\ell_4},\ldots A_{41})$ at i which is quasi-normal (resp. normal) for the first k layers.</u>

<u>Proof</u>: By repeatedly applying Lemma 5.2.3 and Corollary 5.3.12. □

<u>Remark 5.4.6</u>: (i) Assume that $w \in \tilde{A}_n$ has a full DC form $(A_\ell,\ldots,A_1)$ at $i \in \mathbb{Z}$ and assume that there exists $w' = \rho^{A_{i'_1},\ldots,A_{i'_\beta}}_{A_{i_1},\ldots,A_{i_\alpha}}(w)$. Consider the block $A_t(w')$. This is defined up to congruence mod n. But when we mention the u-th entry $e((w'), j^u_t(w'))$ (or $e((w'), j^u_t(w',i')))$ of $A_t(w')$, we mean that the block $A_t(w')$ involved here has been specified precisely by the following rule:

(1) $A_t(w')$ lies between the (i+1)-th row and the (i+n)-th row of w' if $1 \notin \{i_1,\ldots,i_\alpha\} \cup \{i'_1,\ldots,i'_\beta\}$.

(2) $A_t(w')$ lies between the $(i + \sum_{t=1}^{\beta} |A_{i'_t}| + 1)$-th row and the $(i + \sum_{t=1}^{\beta} |A_{i'_t}| + n)$-th row if $1 \in \{i_1,\ldots,i_\alpha\}$.

(3) $A_t(w')$ lies between the $(i - \sum_{t=1}^{\alpha} |A_{i_t}| + 1)$-th row and the $(i - \sum_{t=1}^{\alpha} |A_{i_t}| + n)$-th row if $1 \in \{i'_1,\ldots,i'_\beta\}$.

(ii) Assume that $w \in \tilde{A}_n$ has a full DC form $(A_\ell,\ldots,A_1)$ at $i \in \mathbb{Z}$ and assume that there exists $w'' = \theta^{A_{i'_1},\ldots,A_{i'_\beta}}_{A_{i_1},\ldots,A_{i_\alpha}}(w)$. This time when we mention the u-th entry $e((w''), j^u_t(w''))$ (or $e((w''), j^u_t(w'',i'')))$ of $A_t(w'')$, we specify $A_t(w'')$ involved here by the following rule:

(1) $A_t(w")$ lies between the $(i+1)$-th row and the $(i+n)$-th row of $w"$ if $1 \notin \{i_1,\ldots,i_\alpha\} \cup \{i_1',\ldots,i_\beta'\}$.

(2) $A_t(w")$ lies between the $(i - \sum_{t=1}^{\beta} |A_{i_t'}| + 1)$-th row and the $(i - \sum_{t=1}^{\beta} |A_{i_t'}| + n)$-th row if $1 \in \{i_1,\ldots,i_\alpha\}$.

(3) $A_t(w")$ lies between the $(i + \sum_{t=1}^{\alpha} |A_{i_t}| + 1)$-th row and the $(i + \sum_{t=1}^{\alpha} |A_{i_t}| + n)$-th row if $1 \in \{i_1',\ldots,i_\beta'\}$.

<u>Corollary 5.4.7</u> <u>Assume that $w \in A_n$ has a full MDC form $(A_\ell,\ldots,A_1)$ at $i \in \mathbb{Z}$ which is normal for the first k layers. If there exists $w' = \rho^{A_\ell,\ldots,A_{v+1}}_{A_1,\ldots,A_v}(w)$ for some v, $1 \le v < \ell$, then w' has the full MDC form $(A_\ell,\ldots,A_1)$ at $i + \sum_{t=v+1}^{\ell} |A_t|$ which is normal for the first k layers such that for any h, $1 \le h \le k$,</u>

$$(j_\ell^h(w'), j_{\ell-1}^h(w'),\ldots,j_1^h(w'))^{(om)} = (j_v^h(w), j_{v-1}^h(w),\ldots,j_1^h(w), j_\ell^h(w) + n, j_{\ell-1}^h(w) + n,\ldots,j_{v+1}^h(w) + n)^{(om)} \quad (5.4.8).$$

<u>Proof:</u> Formula (5.4.8) follows from Lemma 5.4.5 since w has an MDC form $(A_v,\ldots,A_1,A_\ell,\ldots,A_{v+1})$ at $i + \sum_{t=v+1}^{\ell} |A_t|$ which is normal for the first k layers. This implies that w' has a full MDC form $(A_\ell,\ldots,A_1)$ at $i + \sum_{t=v+1}^{\ell} |A_t|$ which is normal for the first k layers. □

CHAPTER 6 : THE SUBSET $\sigma^{-1}(\lambda)$ OF THE AFFINE WEYL GROUP $\tilde{A}_n$

In Chapter 3, we have defined a map σ from the affine Weyl group $\tilde{A}_n$ to the set of partitions of n. In this chapter, we consider the fibre $\sigma^{-1}(\lambda)$ corresponding to a given partition λ of n. Our main aim is to show that each element w in this fibre can be transformed by a succession of left star operations into an element y of a rather special form. y has the property that the blocks in a full MDC form have size $\lambda_1,\ldots,\lambda_r$, where $\lambda_1,\ldots,\lambda_r$ are the parts of λ with $\lambda_1 > \ldots > \lambda_r$.

This result will have two useful consequences. In the first place, it enables us to show that the fibre $\sigma^{-1}(\lambda)$ is a union of RL-equivalence classes. This is done in §6.4. In the second place, we shall use this algorithm as the initial stage of a longer process for passing from w to a simpler type of element which will be described in the next chapter.

By Chapter 4, we may assume that $\lambda \neq \{1 > \ldots > 1\}$.

§6.1 TWO SIMPLE LEMMAS ON ITERATED STAR OPERATIONS

Lemma 6.1.1 Assume that $w \in \tilde{A}_n$ has a DC form (A_2,A_1) at $i \in \mathbb{Z}$, where $A_1(w)$ is a longest descending chain in the block $[A_2(w),A_1(w)]$. Then there exists an integer k with $k = \max\{h \mid 1 < h < |A_1|, j_2^1(w) < j_1^h(w)\}$ and $w' = \theta_{A_1}^{A_2}(w)$ such that w' has the DC form (A_1,A_2) at i with $j_2^1(w') = j_1^k(w)$.

<u>Proof</u>: The existence of the element $w' \in \mathbb{A}_n$ and the integer k follows from Lemma 5.3.7 and the condition that $A_1(w)$ is a longest descending chain in the block $[A_2(w), A_1(w)]$. Then formula (5.3.3) and Lemma 5.3.6 imply $j_2^1(w') = j_1^k(w)$. □

<u>Lemma 6.1.2</u> <u>Assume that $w \in \mathbb{A}_n$ has a DC form (A) at $i \in \mathbb{Z}$ with $|A(w)| = \ell$, and suppose that the entry set $\{e(i+\ell+u, j_u(w)) \mid 0 \leq u \leq m\}$ of w with $m+\ell \leq n$ satisfies $j_A^\ell(w) = j_o(w) < j_1(w) < \ldots < j_m(w)$. Then there exists $w' \in \mathbb{A}_n$ with $w \xrightarrow{*(i+1,\ell,m)} w'$ such that w' has the DC form (A) at $i+m$ with $j_A^\ell(w') = j_k(w)$, where $k = \max\{h \mid 0 \leq h \leq m,\ j_h(w) < j_A^{\ell-1}(w)\}$. Let $\{e(i+1+u, j_u(w')) \mid 0 \leq u \leq m\}$ be the entry set of w'. Then $j_o(w') < j_\ell(w') < \ldots < j_m(w') = j_A^\prime(w')$ and $j_h(w') = j_h(w)$ for $0 \leq h \leq k$.</u>

<u>Proof</u>: The existence of w' follows from 5.2.1. The other conclusions follow from Lemma 5.2.3 (ii) and by repeatedly applying formula (5.3.2) with $m_1 = 1$, where m_1 is as in (5.3.2). □

§6.2 <u>THE SUBSET F OF THE AFFINE WEYL GROUP $\mathbb{A}_n$</u>

We shall describe the process of passing from an arbitrary element $w \in \sigma^{-1}(\lambda)$ to an element y of the required form by a succession of left star operations in two stages. In the first stage, we shall show that we can pass from w to an element in the subset F defined below. In the second stage, we shall show that starting from an element of $\sigma^{-1}(\lambda) \cap F$, we can pass to an element y of the required form.

We define $F = \bigcup_{i=0}^{n-1} F_i$ with $F_i = \{w \in \hat{A}_n \mid (i+1)w < \dots < (i+n)w\}$ for $0 \leq i \leq n-1$.

<u>Lemma 6.2.1</u> <u>For any $w \in \hat{A}_n$, there exists $y \in F$ such that $y \underset{P_L}{\sim} w$.</u>

<u>Proof</u>: It suffices to show that there exists y with $y \underset{P_L}{\sim} w$ such that for some $t \in \mathbb{Z}$, we have $(t+1)y < \dots < (t+n)y$. We may assume $w \neq 1$ since otherwise the result is trivial.

By Lemmas 2.2.4 and 2.2.5, there exists $i \in \mathbb{Z}$ such that $(i+1)w < (i+2)w < \dots < (i+j)w$ for some $2 \leq j \leq n$. We apply induction on $\ell = n-j \geq 0$. The result is trivial for $\ell = 0$. Now assume $\ell > 0$. If $(i+j+1)w > (i+j)w$, then the induction can be applied. Otherwise, assume that for some $0 \leq u < j$,

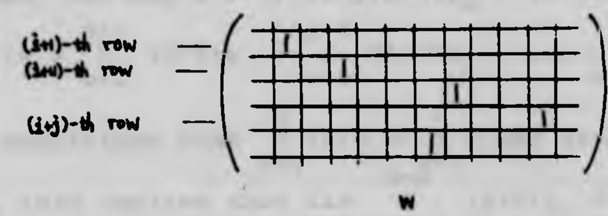
$(i+1)w < (i+2)w < \dots < (i+u)w < (i+j+1)w < (i+u+1)w < \dots < (i+j)w$. Let $z = w$, then $z_o \in \mathcal{D}_L(s_{i+j+1})$. Let $z_1 = {}^*z_o$ in $\mathcal{D}_L(s_{i+j-1})$, then $z_1 \in \mathcal{D}_L(s_{i+j-2})$ if $j \geq 3$. In general, we have $z_u \in \mathcal{D}_L(s_{i+j-u}) \cap \mathcal{D}_L(s_{i+j-u-1})$ for $1 \leq u \leq j-2$, where $z_u = {}^*z_{u-1}$ is in $\mathcal{D}_L(s_{i+j-u})$ for $1 \leq u \leq j-1$. Let $y_1 = z_{j-1}$. Then

$$\begin{cases} (i+v)w = (i+v+1)y_1 & \text{if } 1 \leq v \leq j, \quad v \neq u+1 \\ (i+j+1)w = (i+u+2)y_1 & \\ (i+u+1)w = (i+1)y_1 & \\ (h)w = (h)y_1 & \text{for } \bar{h} \notin \{\overline{i+1}, \overline{i+2}, \dots, \overline{i+j+1}\} \end{cases}$$

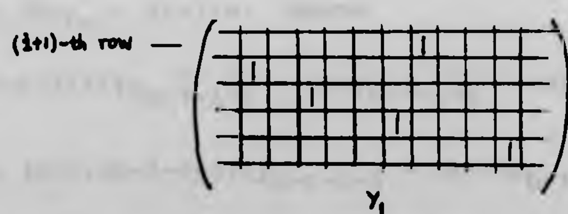
In particular, we have

$$\begin{array}{ccccccccccccc}
(i+2)y_1 & < & (i+3)y_1 & < \dots < & (i+u+1)y_1 & < & (i+u+2)y_1 & < & (i+u+3)y_1 & < \dots < & (i+j+1)y_1 \\
\| & & \| & & \| & & \| & & \| & & \| \\
(i+1)w & < & (i+2)w & < \dots < & (i+u)w & < & (i+j+1)w & < & (i+u+2)w & < \dots < & (i+j)w.
\end{array}$$

Let us take an example to illustrate the above result. Assume that $n > 5$ and w has the form



where $u = 2$ and $j = 4$. Then y_1 has the form



with $(i+1)y_1 = (i+3)w$ and

$$\begin{array}{ccccccc}
(i+2)y_1 & < & (i+3)w & < & (i+4)y_1 & < & (i+5)y_1 \\
\| & & \| & & \| & & \| \\
(i+1)w & < & (i+2)w & < & (i+5)w & < & (i+4)w.
\end{array}$$

Now let us continue to prove Lemma 6.2.1. Clearly, $\sum_{v=2}^{j+1} (i+v)y_1 < \sum_{v=1}^{j} (i+v)w$ and $(i+j+1)y_1 < (i+j)w$. If $(i+j+2)y_1 > (i+j+1)y_1$, then our assertion follows by inductive hypothesis. Otherwise, the same procedure can be repeated.

We get $y_1, y_2, \ldots,$ where for each $c \geq 1$,

$$(i+c+1)y_c < (i+c+2)y_c < \ldots < (i+j+c)y_c, \quad \sum_{v=c+1}^{j+c} (i+v)y_c < \sum_{v=c}^{j+c-1} (i+v)y_{c-1},$$

and $(i+y+c)y_c \leq (i+j)w$. We claim that there exists $c \geq 1$ such that $(i+j+c+1)y_c > (i+j+c)y_c$ and then the induction can be applied. Otherwise, for any $c \geq 1$, $(i+j+c+1)y_c < (i+j+c)y_c$ and then we have

$$\sum_{v=1}^{j} (i+v)w > \sum_{v=2}^{j+1} (i+v)y_1 > \sum_{v=3}^{j+2} (i+v)y_2 > \ldots > \sum_{v=c+1}^{j+c} (i+v)y_c > \ldots .$$

By the conditions that $\sum_{v=1}^{n} (v)x = \sum_{v=1}^{n} v$ and $(v+n)x = (v)x + n$ for any $x \in A_n$, this implies that $\lim_{c\to\infty} \sum_{v=c+1}^{c+n} (i+v)y_c = \infty$. So there exist integers c, h with c sufficiently large and $i+j+c < h \leq i+n+c$ such that $(h)y_c > (i+j)w$. Hence

$$(i+j+(h-1-i-j)+1)y_{h-1-i-j} = (h)y_{h-1-i-j} = (h)y_c > (i+j)w$$

$$\geq (i+j+(h-1-i-j))y_{h-1-i-j} = (h-1)y_{h-1-i-j}.$$

This gives a contradiction. $\square$

For $w \in F_0$, we define an m chain set for w to be a set of m disjoint sequences $\{i_{uv}, 1 \leq v \leq \alpha_u\}$ for $1 \leq u \leq m$ such that $i_{uv} \in \{1,\ldots,n\}$ and i_{uv} are all distinct, and for every u,v with $1 \leq u \leq m$ and $1 \leq v \leq \alpha_u$, we have $(i_{uv})w - (i_{u,v-1})w > n$. We call an m chain set for w saturated if it satisfies the further conditions:

(i) For every u,v with $1 \leq u \leq m$ and $1 \leq v < \alpha_u$,

$$i_{uv} = \max\{h \mid 1 \leq h \leq n,\ (i_{u,v+1})w-(h)w > n,\ h \notin \{i_{pq}, 1\leq p<u, 1\leq q\leq\alpha_p\}\}$$

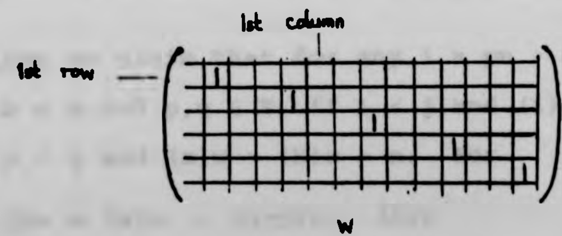
(ii) $\begin{cases} i_{1\alpha_1} = n \\ i_{u\alpha_u} = \max \{h \mid 1 < h < n,\ h \notin \{i_{pq},\ 1 < p < u,\ 1 < q < \alpha_p\}\}, 1<u<m \end{cases}$

(iii) $\{h \mid 1 < h < n,\ (i_{u,1})w-(h)w > n,\ h \notin \{i_{pq}, 1 < p < u,\ 1 < q < \alpha_p\}\} = \emptyset,$
$1<u<m$

(iv) $\sum_{u=1}^{m} \alpha_u = n.$

Clearly, in this case, both m and α_u, $1 < u < m$, are entirely determined by w. So we call such an m chain set the saturated chain set for w and denote it by $N^o(w)$. Obviously, $\alpha_1 > \dots > \alpha_m$, so $N^o(w)$ determines a partition $\{\alpha_1 > \dots > \alpha_m\}$ of n. In such a way, we get a map $F_o \xrightarrow{\phi_o} \Lambda_n$ by sending w to $\{\alpha_1 > \dots > \alpha_m\}$.

For example, assume $n = 5$ and that $w \in F_o$ has the following form:



Then $\{1 < 3 < 5\} \cup \{2 < 4\}$ is the saturated chain set and so $\phi_o(w) = \{3 > 2\} \in \Lambda_5$.

Let Φ be the automorphism of Λ_n such that $\Phi(s_i) = s_{i+1}$ for all $i \in \mathbb{Z}$. Then Φ induces a bijective map from F to itself by restriction to F which sends F_i onto F_{i+1}, $0 < i < n-1$, with the convention that $F_n = F_o$. So we also get a map $F_i \xrightarrow{\phi_i} \Lambda_n$ such that the diagram

$$\begin{tikzcd}
F_i \arrow[rr, "\phi^{n-i}"] \arrow[dr, "\phi_i"'] & \arrow[d, phantom, "\circlearrowright"] & F_o \arrow[dl, "\phi_o"] \\
& \Lambda_n &
\end{tikzcd}$$

commutes.

<u>Lemma 6.2.2</u> <u>For any $w \in F_o \cap \sigma^{-1}(\lambda)$, we have $\phi_o(w) = \lambda$.</u>

<u>Proof</u>: Let $N(w) = \bigcup_{u=1}^{m} \{i_{u1} < i_{u2} < \ldots < i_{u\alpha_u}\}$ be the saturated chain set for w. Let $i'_{uv} = i_{uv} - (v-1)n$ for $1 \le u \le m$, $1 \le v \le \alpha_u$. Then $i'_{u1} > \ldots > i'_{u,\alpha_u}$ and $(i'_{u1})w < \ldots < (i'_{u,\alpha_u})w$ for any $1 \le u \le m$. So $m \ge r$ and $\sum_{t=1}^{\ell} \alpha_t \le \sum_{t=1}^{\ell} \lambda_t$ for any $1 \le \ell \le r$. Now it suffices to show that for any $1 \le \ell \le r$, $\sum_{t=1}^{\ell} \lambda_t \le \sum_{t=1}^{\ell} \alpha_t$.

First we claim that for any $i = pn + a$, $j = qn + b$ with $1 \le a, b \le n$ and $p,q \in \mathbb{Z}$, if $i < j$ and $(i)w > (j)w$, we must have $a > b$, $p < q$ and $(a)w - (b)w > n$. For

$(i)w > (j)w \Rightarrow (a)w > (q-p)n + (b)w$

$\Rightarrow (a)w - (b)w > (q-p)n \ge 0$ since $i < j$

$\Rightarrow (a)w > (b)w$

$\Rightarrow a > b$ by our hypothesis on w

$\Rightarrow p < q$ since $i < j$

$\Rightarrow (a)w - (b)w > n.$

Now suppose $S = S_1 \cup \ldots \cup S_\ell \subset \mathbb{Z}$ satisfies $C_n(w,\ell)$ with $S_t = \{(h_{t1})w > \ldots > (h_{t\beta_t})w \mid h_{t1} < \ldots < h_{t\beta_t}\}$ for $1 \le t \le \ell$ and let h'_{tj} be defined by $\bar{h}'_{tj} = \bar{h}_{tj}$ with $1 \le h'_{th} \le n$. Then, by the above claim, we have $(h'_{tj})w - (h'_{t,j+1})w > n$ and $h'_{tj} > h'_{t,j+1}$ for $1 \le t \le \ell$ and $1 \le j < \beta_t$. Let $k_{tu} = h'_{t,\beta_t+1-u}$ for $1 \le t \le \ell$ and $1 \le u \le \beta_t$. Then $K = \bigcup_{t=1}^{\ell} \{k_{t1} < k_{t2} < \dots < k_{t\beta_t}\}$ is an ℓ chain set for w, and we must show $\sum_{t=1}^{\ell} \beta_t \le \sum_{t=1}^{\ell} \alpha_t$. If n is not a term of any chain of K, then by replacing k_{1,β_1} by n, we also get an ℓ chain set for w.. If n is a term of some chain of K, we may without loss of generality assume that $k_{1\beta_1} = n$. We now define a number h such that $i_{1,\alpha_1+1-j} = k_{1,\beta_1+1-j}$, $1 \le j < h$ for some $1 < h \le \alpha_1 + 1$ with h as large as possible. We shall now modify the set K as follows. If $h \le \alpha_1$, then when i_{1,α_1+1-h} is a term of some chain of K, we replace k_{1,β_1+1-h} by i_{1,α_1+1-h} in the case $\beta_1 \ge h$, or we replace $\{k_{11} < k_{12} < \dots < k_{1h}\}$ by $\{i_{1,\alpha_1+1-h} < i_{1,\alpha_1+2-h} < \dots < i_{1,\alpha_1}\}$ in the case $\beta_1 = h-1$. If $i_{1,\alpha_1+1-h} = k_{uv}$ for some $1 < u \le \ell$, $1 \le v \le \beta_u$, we replace $\{k_{u1} < \dots < k_{uv} < k_{1,\beta_1+2-h} < \dots < k_{1,\beta_1}\}$ by $\{k_{11} < \dots < k_{1,h-1+v}\}$, $\{k_{11} < \dots < k_{1,\beta_1+1-h} < k_{u,v+1} < \dots < k_{u,\beta_u}\}$ by $\{k_{u1} < \dots < k_{u,\beta_1+\beta_u+1-h-v}\}$, $h-1+v$ by β_1 and $\beta_1 + \beta_u +1-h-v$ by β_u, we also get an ℓ chain set for w. If $h = \alpha_1+1$, then by definition of the saturated chain set for w, we have $\beta_1 = \alpha_1$. In such a way, we can make $\beta_1 = \alpha_1$ and replace $\{k_{11} < \dots < k_{1,\beta_1}\}$ by $\{i_{11} < \dots < i_{1\alpha_1}\}$. The cardinal of this new K is equal to or greater than that of the original one. Similarly, we can make $\beta_v = \alpha_v$ and $\{k_{v1} < \dots < k_{v\beta_v}\} = \{i_{v1} < \dots < i_{v\alpha_v}\}$ for all $1 < u \le \ell$. But now we have $|K| = \sum_{t=1}^{\ell} \alpha_t$ and so the cardinal of the initial K

is not greater than $\sum_{t=1}^{\ell} \alpha_t$. Our assertion then follows. □

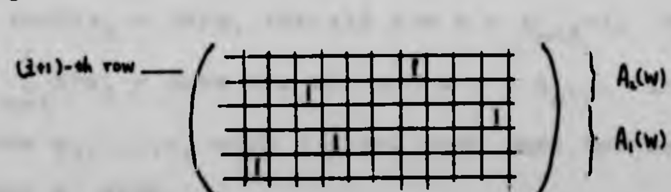
<u>Corollary 6.2.3</u> <u>For any $0 \leq i \leq n-1$, $w \in F_i \cap \sigma^{-1}(\lambda)$, we have $\phi_i(w) = \lambda$.</u>

<u>Proof</u>: $\phi_i(w) = \phi_o(\phi^{n-i}(w))$. Since $w \in \sigma^{-1}(\lambda)$ implies $\phi^{n-i}(w) \in \sigma^{-1}(\lambda)$, our conclusion follows by Lemma 6.2.2. □

§6.3 <u>THE SUBSET H_λ OF $\sigma^{-1}(\lambda)$</u>

Let H_λ be the set of all elements w of $\sigma^{-1}(\lambda)$ which have a full MDC form $(A_r,\ldots,A_1)$ at i for some $i \in \mathbb{Z}$ with $|A_h| = \lambda_h$, $1 \leq h \leq r$. Such an MDC form of $w \in H_\lambda$ is called a standard MDC form of w. By Lemma 2.2.4, $w \in H_\lambda$ implies that $\pi(\mathcal{L}(w)) = \lambda$.

Suppose $n = 5$, $\lambda = \{3 \geq 2\}$. Then the following element $w \in A_5$ lies in the set H_λ.



w

<u>Lemma 6.3.1</u> <u>Let $w \in F \cap \sigma^{-1}(\lambda)$. Then there exist $y \in A_n$ with $y \underset{P_L}{\sim} w$ such that for some $j \in \mathbb{Z}$,</u>

<u>$(j+1)y < (j+2)y < \ldots < (j+n-\lambda_1+1)y > (j+n-\lambda_1+2)y > \ldots > (j+n)y$</u> .

<u>Proof</u>: This is trivial for $w = 1$. Now assume $w \neq 1$. Then we have $(i+n)w - (i+1)w > n$ with $(i+1)w < \ldots < (i+n)w$ for some $0 \le i \le n-1$. Let $i+1 \le i_1 < i_2 < \ldots < i_\alpha = i+n$ such that for each $1 < t \le \alpha$, $(i_t)w - (i_{t-1})w > n$ but $(i_t)w - (i_{t-1}+1)w < n$. By Corollary 6.2.3, we have $\alpha = \lambda_1$. We know that $(i)w > (i+1)w$. By Lemma 6.1.2, there exists $x_1 \in A_n$ with $w \xrightarrow{*(i,2,n-2)} x_1$ satisfying $(i_{\alpha-1})w = (i+n-1)x_1 < (i+n-2)x_1$ and $(i)x_1 < (i+1)x_1 < \ldots < (i+n-2)x_1$ with $(k-1)x_1 = (k)w$ for all $i+1 \le k < i_{\alpha-1}$. If $\alpha \geq 3$, then

$(i)x_1 = (i+1)w < (i_{\alpha-1}-n)w = (i-1)x_1 < (i-2)x_1$. Also by Lemma 6.1.2, there exists $x_2 \in A_n$ with $x_1 \xrightarrow{*(i-2,3,n-3)} x_2$ satisfying

$$(i_{\alpha-2})w = (i_{\alpha-2}-1)x_1 = (i + n - \sum_{t=1}^{2} t)x_2 < (i + n-1 - \sum_{t=1}^{2} t)x_2$$

$$< (i + n-2 - \sum_{t=1}^{2} t)x_2 > (i+n-3 - \sum_{t=1}^{2} t)x_2 > \ldots > (i+1- \sum_{t=1}^{2} t)x_2$$

with $(k-2)x_2 = (k)x_1$ for all $i \le k < i_{\alpha-2}-1$. That is, $(k - \sum_{t=1}^{2} t)x_2 = (k)w$ for all $i+1 \le k < i_{\alpha-2}$. In general, suppose we have $x_1,\ldots,x_\ell$ with $\ell \le \alpha-2$ such that for each $1 < u \le \ell$, the element x_u with

$$x_{u-1} \xrightarrow{*(i+1- \sum_{t=1}^{u} t,\ u+1,\ n-u-1)} x_u$$

satisfies $(i_{\alpha-u})w = (i+n - \sum_{t=1}^{u} t)x_u < (i+n-1- \sum_{t=1}^{u} t)x_u < \ldots <$

$$(i+n-u - \sum_{t=1}^{u} t)x_u > (i+n-u-1- \sum_{t=1}^{u} t)x_u > \ldots > (i+1- \sum_{t=1}^{u} t)x_u \text{ with}$$

$(k - \sum_{t=1}^{u} t)x_u = (k)w$ for all $i + 1 < k < i_{\alpha-u}$.

Then $(i + 1 - \sum_{t=1}^{\ell} t)x_\ell = (i+1)w < (i_{\alpha-\ell}-n)w =$

$(i - \sum_{t=1}^{\ell} t)x_\ell < (i-1 - \sum_{t=1}^{\ell} t)x_\ell < \ldots < (i+1 - \sum_{t=1}^{\ell+1} t)x_\ell$ and

$(i_{\alpha-\ell-1} - \sum_{t=1}^{\ell} t)x_\ell = (i_{\alpha-\ell-1})w$ is the greatest number among the set $\{(v - \sum_{t=1}^{\ell} t)x_\ell \mid i+1 < v < i_{\alpha-\ell}\}$ which is smaller than

$(i_{\alpha-\ell}-n - \sum_{t=1}^{\ell} t)x_\ell = (i_{\alpha-\ell}-n)w$. So by Lemma 6.1.2, there exists

$x_{\ell+1} \in A_n$ with $x_\ell \xrightarrow{*(i+1 - \sum_{t=1}^{\ell+1} t,\ \ell+2,\ n-\ell-2)} x_{\ell+1}$ satisfying

$(i_{\alpha-\ell-1})w = (i_{\alpha-\ell-1} - \sum_{t=1}^{\ell} t)x_\ell = (i+n-\sum_{t=1}^{\ell+1} t)x_{\ell+1} < (i+n-1-\sum_{t=1}^{\ell+1} t)x_{\ell+1} < \ldots$

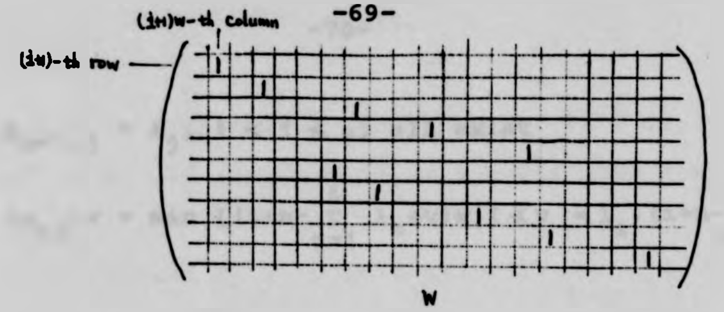
$< (i+n-\ell-1 - \sum_{t=1}^{\ell+1} t)x_{\ell+1} > (i+n-\ell-2-\sum_{t=1}^{\ell+1} t)x_{\ell+1} > \ldots > (i+1-\sum_{t=1}^{\ell+1} t)x_{\ell+1}$

with $(k - \sum_{t=1}^{\ell+1} t)x_{\ell+1} = (k)w$ for all $i+1 < k < i_{\alpha-\ell-1}$. Clearly, if $\ell \leq \alpha-3$, then $(i+1 - \sum_{t=1}^{\ell+1} t)x_{\ell+1} = (i+1)w < (i_{\alpha-\ell-1}-n)w$

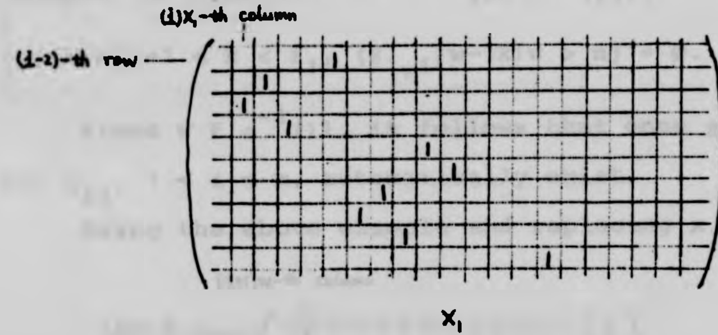
$= (i - \sum_{t=1}^{\ell+1} t)x_{\ell+1} < (i-1-\sum_{t=1}^{\ell+1} t)x_{\ell+1} < \ldots < (i+1-\sum_{t=1}^{\ell+2} t)x_{\ell+1}$

and the same procedure can be repeated. Finally, we get $y = x_{\alpha-2}$ and take $j = i - \sum_{t=1}^{\alpha-1} t$. Then y has the required property. □

Suppose $n = 5$ and that $w \in A_5$ is the following matrix.



Then $w \in F \cap \sigma^{-1}(\{3 > 2\})$. There exists x_1 with $w \xrightarrow{*(i,2,3)} x_1$ which is as follows.



with $(i)x_1 = (i+1)w$. So x_1 here plays the role of y in Lemma 6.3.1.

Assume $w \in \sigma^{-1}(\lambda)$ such that for some $i,\ m \in \mathbb{Z}$ with $1 < m < r$, we have

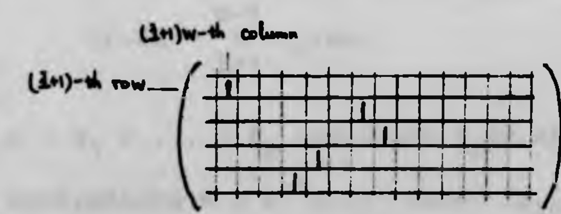
$$\begin{cases} (i+1)w < \ldots < (i+n+1-\sum_{t=1}^{m}\lambda_t)w \\ (i+n+1-\sum_{t=1}^{\ell}\lambda_t)w > \ldots > (i+n-\sum_{t=1}^{\ell-1}\lambda_t)w,\ 1 < \ell < m \end{cases}$$

Let $i+1 < i_1 < i_2 < \ldots < i_\alpha = i+n-\sum_{t=1}^{m}\lambda_t$ such that

$$\begin{cases} \beta_{m+1,j} = i_j, \ 1 < j < \alpha, \ \text{all exist} \\ (\beta_{\ell j})w = \min \{(i+n-\sum\limits_{t=1}^{\ell} \lambda_t+v)w \mid 1 < v < \lambda_\ell, (i+n-\sum\limits_{t=1}^{\ell} \lambda_t+v)w > (\beta_{\ell+1,j})w\}, & 1 < j < \alpha \\ & 1 < \ell < m \\ (i_j)w = \max \{(k)w \mid i+1 < k < i_{j+1}, \ (\beta_{1,j+1})w-(k)w > n\}, & 1 < j < \alpha \\ \{(k)w \mid i+1 < k < i_1, \ (\beta_{1,1})w-(k)w > n\} = \emptyset. \end{cases}$$

Since $w \in \sigma^{-1}(\lambda)$, it follows that once $\beta_{m+1,j}$ exists, all $\beta_{\ell j}$, $1 < \ell < m$, automatically exist.

Using the above example and replacing x_1 by w, we have



w

with $m = 1$ and $\alpha = 2$. Thus $\beta_{2,1} = i+1$, $\beta_{2,2} = i+2$, $\beta_{1,1} = i+5$ and $\beta_{1,2} = i+3$.

We shall prove that

<u>Lemma 6.3.2</u> <u>Let $w \in \sigma^{-1}(\lambda)$ be as above. Then $\alpha = \lambda_{m+1}$.</u>

Before doing this, let us show some simple results.

<u>Lemma 6.3.3</u> <u>Let $w \in \sigma^{-1}(\lambda)$ be as in Lemma 6.3.2. Then</u>

<u>$0 < (\beta_{1j})w - (i_j)w < n$ for any $1 < j < \alpha$.</u>

<u>Proof</u>: It suffices to show that $(\beta_{1j})w - (i_j)w < n$ for any $1 < j < \alpha$. If $(\beta_{1j})w - (i_j)w > n$ for some $1 < j < \alpha$, let

$$S_u = \{(i+n+1 - \sum_{t=1}^{u} \lambda_t)w > (i+n+2 - \sum_{t=1}^{u} \lambda_t)w > \dots >$$

$$(\beta_{uj})w > (\beta_{u-1,j}+1)w > \dots > (i+n-\sum_{t=1}^{u-2} \lambda_t)w\} \text{ for } 1 < u < m$$

$$S_1 = \{(i+n+1 - \lambda_1)w > (i+n+2-\lambda_1)w > \dots >$$

$$(\beta_{1j})w > (i_j+n)w > (\beta_{mj}+1+n)w > (\beta_{mj}+2+n)w > \dots >$$

$$(i+2n - \sum_{t=1}^{m-1} \lambda_t)w\}.$$

Then $S = S_1 \cup \dots \cup S_m$ satisfies $C_n(w,m)$ with $|S| = \sum_{t=1}^{m} \lambda_t+1$. This contradicts $w \in \sigma^{-1}(\lambda)$. Also, $(\beta_{1j})w - (i_j)w \neq n$ since $\overline{(\beta_{1j})w} \neq \overline{(i_j)w}$. Our result follows. □

<u>Corollary 6.3.4 Let $w \in \sigma^{-1}(\lambda)$ be as in Lemma 6.3.2. Then for $1 < \ell, \ell' < m$ and $1 < j, j' < \alpha$, we have</u>

$$\underline{\beta_{\ell j} = \beta_{\ell' j'} \iff \ell = \ell' \text{ and } j = j'}.$$

<u>Proof</u>: ($\Leftarrow$) This is obvious.

($\Rightarrow$) It is clear that $\beta_{\ell j} = \beta_{\ell' j'}$ implies $\ell = \ell'$. So it suffices to show that for any $1 < \ell < m$ and $1 < j < \alpha$, $(\beta_{\ell j})w < (\beta_{\ell,j+1})w$ holds. In general, we have $(\beta_{\ell j})w < (\beta_{\ell,j+1})w$.

If $(\beta_{\ell j})w = (\beta_{\ell,j+1})w$ for some $1 < \ell < m$, $1 < j < \alpha$, then we have $(\beta_{1j})w = (\beta_{1,j+1})w$. But $(\beta_{1j})w - (i_j)w < n$ and $(\beta_{1,j+1})w - (i_j)w > n$. This gives a contradiction. □

<u>Lemma 6.3.5</u> <u>Let $w \in \sigma^{-1}(\lambda)$ be as in Lemma 6.3.2. Then $\alpha < \lambda_{m+1}$.</u>

<u>Proof</u>: We shall define sets $S_1, S_2, \ldots, S_m, S_{m+1}$ as follows. For $1 < u < m$, let

$$S_u = \{(i+n+1-\sum_{t=1}^{u}\lambda_t)w > (i+n+2-\sum_{t=1}^{u}\lambda_t)w > \ldots > (\beta_{u\alpha})w$$

$$> (\beta_{u-1,\alpha}+1)w > (\beta_{u-1,\alpha}+2)w > \ldots > (\beta_{k,\alpha-u+k})w$$

$$> (\beta_{k-1,\alpha-u+k}+1)w > (\beta_{k-1,\alpha-u+k})w > \ldots > (\beta_{k',\alpha-u+k'})w$$

$$> (\beta_{k'-1,\alpha-u+k'-1})w > (\beta_{k'-2,\alpha-u+k'-1}+1)w > (\beta_{k'-2,\alpha-u+k'-1}+2)w > \ldots\}$$

$$S_{m+1} = \{(\beta_{m+1,\alpha})w > (\beta_{m,\alpha}+1)w > (\beta_{m,\alpha}+2)w > \ldots > (\beta_{k,\alpha-m-1+k})w$$

$$> (\beta_{k-1,\alpha-m-1+k}+1)w > (\beta_{k-1,\alpha-m-1+k}+2)w > \ldots > (\beta_{k',\alpha-m-1+k'})w$$

$$> (\beta_{k'-1,\alpha-m-1+k'-1})w > (\beta_{k'-2,\alpha-m-1+k'-1}+1)w > (\beta_{k'-2,\alpha-m-1+k'-1}+2)w > \ldots\}$$

where $k \not\equiv 1$, $k' \equiv 1 \pmod{m+1}$ and $\beta_{f,g} = \beta_{f+m+1,g+m+1} + n$. It remains to specify the last term of each S_v. Let $\alpha - v = k(m+1) + q$, $1 < q < m+1$, $h \in \mathbb{Z}$. Then the last term of S_v, $1 < v < m+1$, is

$$\begin{cases} (i+(h+2)n - \sum\limits_{t=1}^{m-q} \lambda_t)w \text{ if } \beta_{m+1-q,1} \neq i+n-\sum\limits_{t=1}^{m-q} \lambda_t \text{ and } 1 < q < m. \\ (\beta_{m+2-q,1} + (h+1)n)w \text{ otherwise.} \end{cases}$$

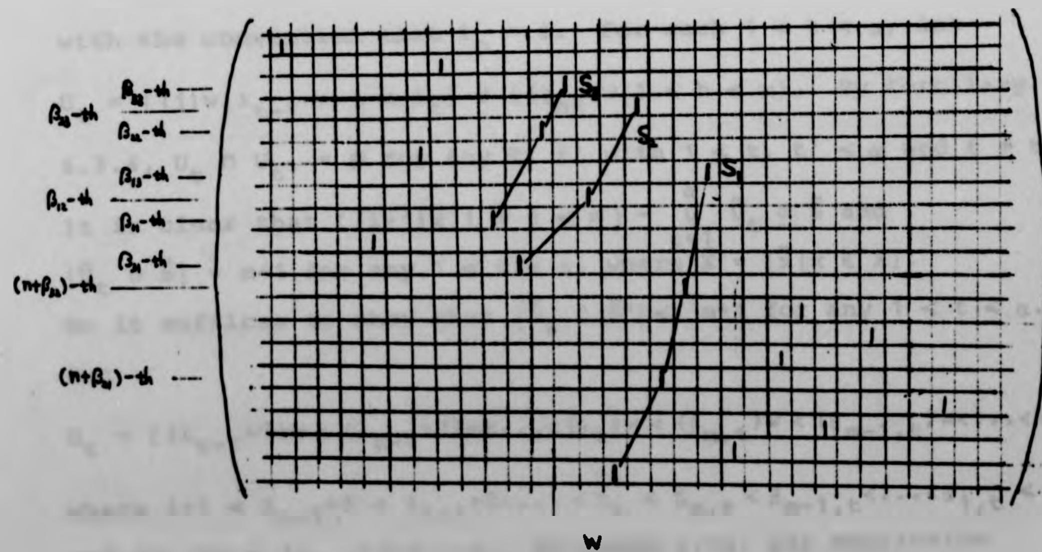
Then $S = S_1 \cup \ldots \cup S_{m+1}$ satisfies $C_n(w,m+1)$ with

$$\bar{S} = \{\overline{(i+n+1-\sum_{t=1}^{m} \lambda_t)w},\ \overline{(i+n+2-\sum_{t=1}^{m} \lambda_t)w}, \ldots, \overline{(i+n)w},\ \overline{i_j},\ 1 < j < \alpha\}.$$

It follows that $|S| = \sum_{t=1}^{m} \lambda_t + \alpha$. So $\alpha < \lambda_{m+1}$ by $w \in \sigma^{-1}(\lambda)$. □

An example of $S = S_1 \cup \ldots \cup S_{m+1}$ in the proof of Lemma 6.3.5 is as follows.

Suppose $n = 10$, $\lambda = \{4 > 3 > 3\}$ and that $w \in \sigma^{-1}(\lambda)$ is as below.



Then w is as in Lemma 6.3.5 with $m = 2$ and $\alpha = 3$. The sets S_1, S_2, S_3 are as in this diagram which consist of all entries of w occurring as the vertices of the corresponding broken lines. We see that the lowest vertex of each broken line is either the last term of some maximal descending chain of w or lies in some congruent row of the $\beta_{\ell,j}$-th row of w for some j,ℓ with $1 < j < \alpha$ and $1 < \ell < m+1$.

Proof of Lemma 6.3.2

Let $S = S_1 \cup \ldots \cup S_{m+1}$ be as in the proof of Lemma 6.3.5. It suffices to show that if $S' = S'_1 \cup \ldots \cup S'_{m+1}$ satisfies $C_n(w,m+1)$, then $|S'| < \sum_{t=1}^{m} \lambda_t + \alpha$. We know that

$$\{(i+1)w,\ (i+2)w, \ldots, (i+n-\sum_{t=1}^{m} \lambda_t)w\} = \bigcup_{1<t<\alpha} \{(j)w \mid i_{t-1} < j < i_t\}$$

with the convention that $i_o = i$. For each $1 < t < \alpha$, let $U_t = \{(j)w \mid i_{t-1} < j < i_t\} \cup \{(\beta_{ht})w \mid 1 < h < m\}$. By Corollary 6.3.4, $U_t \cap U_{t'} = \emptyset$ for any t, t' with $1 < t, t' < \alpha$ and $t \neq t'$. It is clear that $\{\overline{(i+j)w} \mid 1 < j < n\} - \bigcup_{t=1}^{\alpha} \bar{U}_t \subseteq \bar{S}$ and $|\bar{U}_t \cap \bar{S}| = m+1$ for any $1 < t < \alpha$, where $\bar{X} = \{\bar{x} \mid x \in X\}$. So it suffices to show that $|\bar{U}_t \cap \bar{S}'| < m+1$ for any $1 < t < \alpha$. But

$$U_t = \{(i_{t-1}+1)w < (i_{t-1}+2)w < \ldots < (i_t)w < (\beta_{m,t})w < (\beta_{m-1,t})w < \ldots < (\beta_{1,t})w\}$$

where $i+1 < i_{t-1}+1 < i_{t-1}+2 < \ldots < i_t < \beta_{m,t} < \beta_{m-1,t} < \ldots < \beta_{1,t} < i+n$ and $(\beta_{1t})w - (i_{t-1}+1)w < n$. By Lemma 3.10, our conclusion follows. □

<u>Lemma 6.3.6 Let $w \in \sigma^{-1}(\lambda)$ be as in Lemma 6.3.2. Then there exists $y \in A_n$ with $y \underset{P_L}{\sim} w$ such that for some $j \in \mathbb{Z}$,</u>

$$\begin{cases} \underline{(j+1)w < (j+2)w < \ldots < (j+n+1-\sum_{t=1}^{m+1} \lambda_t)w} \\ \underline{(j+n+1-\sum_{t=1}^{\ell} \lambda_t)w > \ldots > (j+n-\sum_{t=1}^{\ell-1} \lambda_t)w, \quad 1 \leq \ell \leq m+1} \end{cases}$$

<u>Proof</u>: If $\alpha = 1$, then $y = w$ is as required. Now assume $\alpha > 1$. By Lemma 5.2.2, there exists a sequence of elements $x_{10} = w$, $x_{11},\ldots,x_{1m}$ such that for each $1 \leq \ell \leq m$, we have $x_{1\ell} \xleftarrow{*(i+n+1-\sum_{t=1}^{m+1-\ell} \lambda_t, \lambda_{m+1-\ell})} x_{1,\ell-1}$. By Lemma 6.1.1, $(i)x_{1m} = (\beta_{1\alpha})w-n > (i_{\alpha-1})w > (i+1)w = (i+1)x_{1m}$. So by Lemma 6.1.2, there exists $x_{1,m+1}$ with

$x_{1m} \xrightarrow{*(i,2,n-\sum_{t=1}^{m}\lambda_t-2)} x_{1,m+1}$ having the following properties:

(i) $(i_{\alpha-1})w = (i+n-1-\sum_{t=1}^{m} \lambda_t)x_{1,m+1} > (i+n-2-\sum_{t=1}^{m} \lambda_t)x_{1,m+1}$.

(ii) $(i+n-2-\sum_{t=1}^{m} \lambda_t)x_{1,m+1} > (i+n-3-\sum_{t=1}^{m} \lambda_t)x_{1,m+1} > \ldots > (i)x_{1,m+1}$

with $(k-1)x_{1,m+1} = (k)w$ for all k with $i+1 \leq k \leq i_{\alpha-1}$ or $\beta_{\ell\alpha} \leq k \leq i+n-\sum_{t=1}^{\ell-1} \lambda_t$, $1 \leq \ell \leq m$.

(iii) $(i+n-\sum_{t=1}^{\ell} \lambda_t)x_{1,m+1} > \ldots > (i+n-1-\sum_{t=1}^{\ell-1} \lambda_t)x_{1,m+1}$ for $1 \leq \ell \leq m$.

If $\alpha > 2$, then there exists elements $x_{20} = x_{1,m+1}$, $x_{21},\ldots,x_{2m}$ such that for each $1 < \ell < m$, we have $x_{2\ell} \xleftarrow{*(i+n-\sum_{t=1}^{m+1-\ell} \lambda_t, \lambda_{m+1-\ell}, 2)} x_{2,\ell-1}$. By Lemma 6.1.1,

$$(i-2)x_{2m} > (i-1)x_{2m} = (\beta_{1,\alpha-1})w-n > (i_{\alpha-2})w > (i+1)w = (i)x_{1,m+1} = (i)x_{2,m}.$$

So by Lemma 6.1.2, there exists $x_{2,m+1}$ with $x_{2,m} \xrightarrow{*(i-2,3,n-\sum_{t=1}^{m}\lambda_t-3)} x_{2,m+1}$ satisfying the properties as follows:

(i) $(i_{\alpha-2})w = (i+n-3-\sum_{t=1}^{m} \lambda_t)x_{2,m+1} < (i+n-4-\sum_{t=1}^{m} \lambda_t)x_{2,m+1}$

$$< (i+n-5-\sum_{t=1}^{m} \lambda_t)x_{2,m+1}.$$

(ii) $(i+n-5-\sum_{t=1}^{m} \lambda_t)x_{2,m+1} > (i+n-6-\sum_{t=1}^{m} \lambda_t)x_{2,m+1} > \ldots > (i-2)x_{2,m+1}$

with $(k-\sum_{t=1}^{2} t)x_{2,m+1} = (k-1)x_{1,m+1} = (k)w$ for all k with $i+1 < k < i_{\alpha-2}$ or $\beta_{\ell,\alpha-1} < k < i+n-\sum_{t=1}^{\ell-1} \lambda_t$, $1 < \ell < m$.

(iii) $(i+n-2-\sum_{t=1}^{\ell} \lambda_t)x_{2,m+1} > \ldots > (i+n-3-\sum_{t=1}^{\ell-1} \lambda_t)x_{2,m+1}$ for $1 < \ell < m$.

In general, if we have got $x_{h,m+1}$, $h < \alpha-1$, satisfying

(i) $(i_{\alpha-h})w = (i+n-\sum_{t=1}^{h} t - \sum_{t=1}^{m} \lambda_t)x_{h,m+1} < (i+n-1-\sum_{t=1}^{h} t - \sum_{t=1}^{m} \lambda_t)x_{h,m+1}$

$$< \ldots < (i+n-h-\sum_{t=1}^{h} t - \sum_{t=1}^{m} \lambda_t)x_{h,m+1}.$$

(ii) $(i+n-h-\sum_{t=1}^{h} t-\sum_{t=1}^{m}\lambda_t)x_{h,m+1} > (i+n-h-1-\sum_{t=1}^{h} t-\sum_{t=1}^{m}\lambda_t)x_{h,m+1}$

$> \ldots > (i+1-\sum_{t=1}^{h}.t)x_{h,m+1}$ with $(k-\sum_{t=1}^{h} t)x_{h,m+1} = k(w)$ for all

k with $i+1 \le k < i_{\alpha-h}$ or $\beta_{\ell,\alpha-h+1} < k \le i+n-\sum_{t=1}^{\ell-1}\lambda_t$, $1 \le \ell \le m$.

(iii) $(i+n+1-\sum_{t=1}^{h} t-\sum_{t=1}^{\ell}\lambda_t)x_{h,m+1} > \ldots > (i+n-\sum_{t=1}^{h} t-\sum_{t=1}^{\ell-1}\lambda_t)x_{h,m+1}$

for $1 \le \ell \le m$. Then there exists elements $x_{h+1,o} = x_{h,m+1}$, $x_{h+1,1},\ldots,x_{h+1,m}$ such that for each $1 \le \ell \le m$, we have

$$x_{h+1,\ell} \xleftarrow{{}^*(i+n+1-\sum_{t=1}^{h} t-\sum_{t=1}^{m+1-\ell}\lambda_t,\ \lambda_{m+1-\ell},\ h+1)} x_{h+1,\ell-1}.$$

By Lemma 6.1.1, we see that

$(i+1-\sum_{t=1}^{h+1} t)x_{h+1,m} > (i+2-\sum_{t=1}^{h+1} t)x_{h+1,m} > \ldots > (i-\sum_{t=1}^{h} t)x_{h+1,m}$

$= (\beta_{1,\alpha-h})w-n > (i_{\alpha-h-1})w \ge (i+1)w = (i+1-\sum_{t=1}^{h} t)x_{h+1,m}$. So

by Lemma 6.1.2, there exists $x_{h+1,m+1}$ with

$$x_{h+1,m} \xrightarrow{{}^*(i+1-\sum_{t=1}^{h+1}t,\ h+2,\ n-\sum_{t=1}^{m}\lambda_t-h-2)} x_{h+1,m+1}$$

satisfying the properties as below:

(i) $(i_{\alpha-h-1})w = (i+n-\sum_{t=1}^{h+1} t-\sum_{t=1}^{m}\lambda_t)x_{h+1,m+1} < (i+n-1-\sum_{t=1}^{h+1} t-$
$\sum_{t=1}^{m}\lambda_t)x_{h+1,m+1} < \ldots < (i+n-h-1-\sum_{t=1}^{h+1} t-\sum_{t=1}^{m}\lambda_t)x_{h+1,m+1}$.

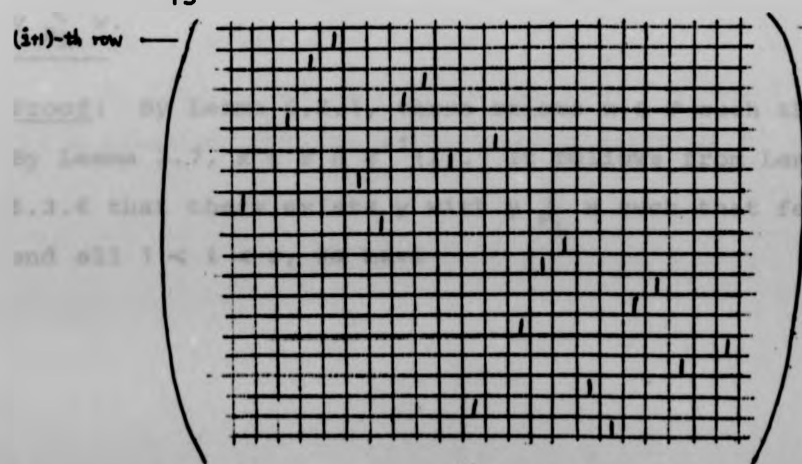
(ii) $(i+n-h-1-\sum_{t=1}^{h+1} t - \sum_{t=1}^{m} \lambda_t)x_{h+1,m+1} > (i+n-h-2-\sum_{t=1}^{h+1} t - \sum_{t=1}^{m} \lambda_t)x_{h+1,m+1} > \dots > (i+1-\sum_{t=1}^{h+1} t)x_{h+1,m+1}$ with $(k - \sum_{t=1}^{h+1} t)x_{h+1,m+1} = (k)w$ for all k with $i+1 < k < i_{\alpha-h-1}$ or $\beta_{\ell,\alpha-h} < k < i+n - \sum_{t=1}^{\ell-1} \lambda_t$, $1 < \ell < m$.

(iii) $(i+n+1-\sum_{t=1}^{h+1} t - \sum_{t=1}^{\ell} \lambda_t)x_{h+1,m+1} > \dots > (i+n - \sum_{t=1}^{h+1} t - \sum_{t=1}^{\ell-1} \lambda_t)x_{h+1,m+1}$ for $1 < \ell < m$. So if $h + 1 < \alpha - 1$, such a recurring procedure can be carried on. Finally, we get $y = x_{\alpha-1,m+1}$. By Lemma 6.3.2, y is as required. □

As in the example for the proof of Lemma 6.3.5, let $x_{10} = w$, x_{11}, x_{12}, x_{13} be a sequence such that

$$x_{11} \xleftarrow{{}^*(i+4,\lambda_2)} x_{10}, \quad x_{12} \xleftarrow{{}^*(i+7,\lambda_1)} x_{11}, \quad x_{12} \xrightarrow{{}^*(i,2)} x_{13}.$$

We have x_{13} as follows:

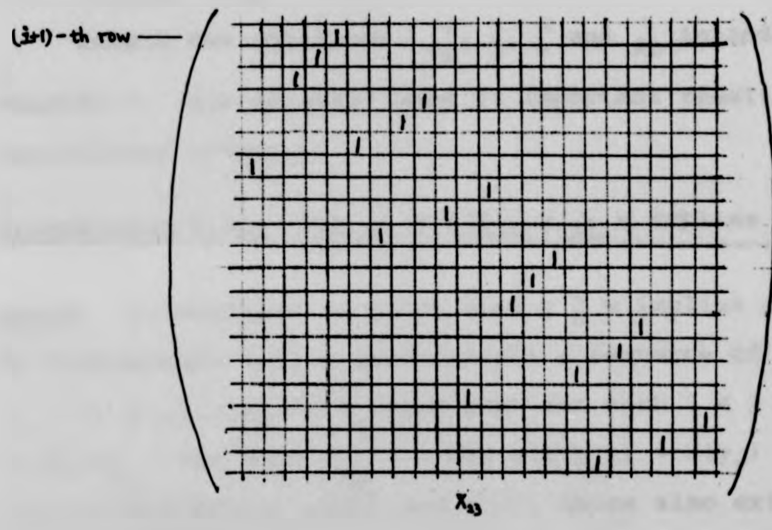


X_{13}

Then let $x_{20} = x_{13}, x_{21}, x_{22}, x_{23}$ be a sequence such that

$$x_{21} \xleftarrow{*(i+3,\lambda_2,2)} x_{20},\quad x_{22} \xleftarrow{*(i+6,\lambda_1,2)} x_{21},\quad x_{22} \xrightarrow{*(i-2,3,0)} x_{23}.$$

We get the following x_{23}



which is required.

Now we can prove our main result in this chapter.

<u>Proposition 6.3.7</u> <u>For any $w \in \sigma^{-1}(\lambda)$, there exists $y \in H_\lambda$ with $y \underset{P_L}{\sim} w$.</u>

<u>Proof</u>: By Lemma 6.2.1, there exists $x \in F$ such that $x \underset{P_L}{\sim} w$. By Lemma 3.7, $x \in F \cap \sigma^{-1}(\lambda)$. It follows from Lemmas 6.3.1 and 6.3.6 that there exists y with $y \underset{P_L}{\sim} x$ such that for some $j \in \mathbb{Z}$ and all $1 < \ell < r$, we have

$(j+1+\sum_{t=\ell+1}^{r}\lambda_t)y > (j+2+\sum_{t=\ell+1}^{r}\lambda_t)y > \dots > (j+\sum_{t=\ell}^{r}\lambda_t)y$. i.e.

$y \in H_\lambda$. □

§6.4 $\sigma^{-1}(\lambda)$ IS A UNION OF RL-EQUIVALENCE CLASSES

Recall the notations $\underset{\Gamma_W}{\sim}$, $\underset{L}{\sim}$, $\underset{R}{\sim}$ and $\underset{RL}{\sim}$ introduced in Chapter 1. Now we shall show an important result on the RL-equivalence relation.

Proposition 6.4.1 For $y, w \in A_n$, $y \underset{RL}{\sim} w$ implies $\sigma(y) = \sigma(w)$.

Proof: It suffices to prove that $y \underset{R}{\sim} w$ implies $\sigma(y) = \sigma(w)$. By Proposition 6.3.7, there exists a sequence of elements $y_0 = y, y_1, \dots, y_r$ in A_n such that for each $1 \le j \le r$, $y_j = {}^*y_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ for some $s_{i_j} \in \Delta$ and $\pi(\mathcal{L}(y_r)) = \sigma(y_r) = \sigma(y)$. Since $w \underset{R}{\sim} y$, by Theorems A(ii) and C(i), there also exists a sequence of elements $w_0 = w, w_1, \dots, w_r$ in A_n such that for each $1 \le j \le r$, $w_j = {}^*w_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ and for each $0 \le k \le r$, $\mathcal{L}(w_k) = \mathcal{L}(y_k)$. In particular, $\pi(\mathcal{L}(w_r)) = \pi(\mathcal{L}(y_r)) = \sigma(y)$. By Lemma 3.6, $\sigma(w) = \sigma(w_r) \geq \pi(\mathcal{L}(w_r)) = \sigma(y)$. By symmetry, $\sigma(y) \geq \sigma(w)$. Therefore, we get $\sigma(y) = \sigma(w)$. □

By Proposition 6.4.1, we see that for any $\lambda \in \Lambda_n$, $\sigma^{-1}(\lambda)$ is a union of some RL-equivalence classes of A_n.

CHAPTER 7 : THE SET $\mathbb{N}_\lambda$ OF NORMALIZED ELEMENTS OF $\sigma^{-1}(\lambda)$

In Chapter 6, we have shown that any element $w \in \sigma^{-1}(\lambda)$ is P_L- equivalent to some element of H_λ. Now we shall prove that any element $w \in H_\lambda$ is P_L-equivalent to some normalized element defined below. A normalized element has a quite simple form and good properties which will be useful to us later.

In §6.3, we have introduced the set H_λ. Here we give a lemma for H_λ. Let $\lambda = \{\lambda_1 \geq \ldots \geq \lambda_r\} \in \Lambda_n$ and let $\mu = \{\mu_1 \geq \ldots \geq \mu_m\} \in \Lambda_n$ be the dual partition of λ.

<u>Lemma 7.1</u> <u>Assume that $w \in H_\lambda$ has a standard MDC form $(A_r,\ldots,A_1)$ at $i \in \mathbb{Z}$. Then for any $1 \leq t \leq \lambda_1$,</u>

$$\underline{j^t_{\mu_t}(w) < j^t_{\mu_t-1}(w) < \ldots < j^t_1(w)}$$

<u>Proof</u>: Otherwise, there exists some t, h with $1 \leq t \leq \lambda_1$ and $1 \leq h < \mu_t$ such that $j^t_h(w) < j^t_{h+1}(w)$. Let

$$S_v = \{e((w),\ j^u_v(w))\,|\,1 \leq u \leq \lambda_v\},\quad 1 \leq v < h$$

$$S_h = \{e((w),\ j^u_{h+1}(w)),\ e((w),\ j^u_h(w))\,|\,1 \leq u \leq t,\ t \leq u' \leq \lambda_h\}$$

Then $S = S_1 \cup \ldots \cup S_h$ satisfies $C_n(w,h)$ with $|S| = \sum_{u=1}^{h} \lambda_u + 1$. This contradicts $w \in \sigma^{-1}(\lambda)$. So our proof is complete. □

<u>Definition 7.2</u> Let $\mathbb{N}_\lambda$ be the set of all elements $w \in H_\lambda$, called normalized elements of $\sigma^{-1}(\lambda)$, such that w has a standard MDC

form $(A_r,\ldots,A_1)$ at $i \in \mathbb{Z}$ which is normal. In that case, if for any $1 \le u \le \lambda_1$, either $j_t^u(w) > j_{t+1}^{u-1}(w)$ for some $1 \le t < \mu_u$ or $j_{\mu_u}^u(w) > j_1^{u-1}(w)-n$, then w is called a principal normalized element of $\sigma^{-1}(\lambda)$. Let $\bar{N}_\lambda$ be the set of all principal normalized elements of $\sigma^{-1}(\lambda)$.

Strictly speaking, we shall call an element defined above a left normalized element (or a left principal normalized element), since we can also define a right normalized element (or a right principal normalized element) to be an element which is obtained by transposing a left normalized element (or a left principal normalized element). But in this thesis, we shall only consider left normalized elements and left principal normalized elements. So the word "left" here is usually omitted.

Now we shall give some examples of elements in N_λ or in $\bar{N}_\lambda$.

(i) Assume that $J^i = J_1^i \cup \ldots \cup J_r^i \in \underline{\Delta}$ with $J_j^i = \{s_{\alpha_j+1}, s_{\alpha_j+2},\ldots,s_{\alpha_j+\lambda_j-1}\}$, $1 \le j \le r$ and $\alpha_j = i + \sum\limits_{h=j+1}^{r} \lambda_h$ for some $i \in \mathbb{Z}$. Let $w_o^{J^i}$ be the longest element in W_{J^i}. Then by Lemma 3.11, we have $w_o^{J^i} \in \sigma^{-1}(\lambda)$. So by observing its matrix, we can see that $w_o^{J^i}$ has a full MDC form $(A_r,A_{r-1},\ldots,A_1)$ at i with $|A_j(w_o^{J^i})| = \lambda_j$ for any $1 \le j \le r$. We also see that the t-th entry of $A_h(w_o^{J^i})$ is $e((w_o^{J^i}), j_h^t(w_o^{J^i})) = e(i + \sum\limits_{u=h+1}^{r} \lambda_u + t,$ $i + \sum\limits_{u=h}^{r} \lambda_u + 1 - t)$ for any h, t with $1 \le h \le r$ and $1 \le t \le \lambda_h$. We can check that for any $1 \le t \le \lambda_1$, $j_1^t(w_o^{J^i}) - n < j_{\mu_t}^t(w_o^{J^i}) < j_{\mu_t-1}^t(w_o^{J^i}) < \ldots < j_1^t(w_o^{J^i})$. Moreover, for $1 \le t \le \lambda_1$, if $\mu_t \ge 2$, then we have $j_1^{t+1}(w_o^{J^i}) = i + \sum_{u=1}^{r} \lambda_u + 1 - (t+1) > i + \sum_{u=2}^{r} \lambda_u + 1-t = j_2^t(w_o^{J^i})$; if $\mu_t = 1$, then we have $j_1^{t+1}(w_o^{J^i}) = j_1^t(w_o^{J^i})-1 > j_1^t(w_o^{J^i})-n$. So $w_o^{J^i} \in \bar{N}_\lambda$. The elements $w_o^{J^i}$, $0 \le i < n$, in $\bar{N}_\lambda$ will play an important role in Chapter 11.

(ii) If $\lambda = \{n\}$, then we have $N_\lambda = H_\lambda$. An element $w \in N_\lambda$ lies in $\bar{N}_\lambda$ if and only if w has a standard MDC form (A) at $i \in \mathbb{Z}$ with $j_A^t(w) - j_A^{t+1}(w) < n$ for all $1 \le t < n$.

We know that for any $1 \le t, t' \le n$ with $t \ne t'$, $\overline{j_A^t(w)} \ne \overline{j_A^{t'}(w)}$. On the other hand, if we are given a permutation of $\bar{1}, \bar{2}, \dots, \bar{n}$, say $\bar{i}_1, \bar{i}_2, \dots, \bar{i}_n$, then there exists a unique element $w \in \tilde{A}_n$ which has an MDC form (A) at i for some $i \in \mathbb{Z}$ with $|A(w)| = n$ such that $\overline{j_A^u(w)} = \bar{i}_u$ and $j_A^t(w) - j_A^{t+1}(w) < n$ for all u,t with $1 \le u \le n$ and $1 \le t < n$. This implies that $|\bar{N}_{\{n\}}| = n!$. Later on, we shall see that $|\bar{N}_{\{n\}}|$ is just the number of left cells in $\sigma^{-1}(\{n\})$. But this result cannot be extended to the general case when $\lambda \in \Lambda_n$ is arbitrary.

(iii) If $\lambda = \{2 \ge 1 \ge \dots \ge 1\}$, then $H_\lambda = \sigma^{-1}(\lambda)$. Suppose $w \in \sigma^{-1}(\lambda)$ with $s_t \in \mathcal{L}(w)$. Then $w \in N_\lambda$ if and only if $(t+2)w > (t)w$. Also, $w \in \bar{N}_\lambda$ if and only if $w \in N_\lambda$ with $\ell(w) < n$.

<u>Lemma 7.3</u> <u>For $\ell \ge 2$, assume that $w \in \sigma^{-1}(\lambda)$ has an MDC form $(A_{\ell-1}, \dots, A_1, A_\ell)$ at $i \in \mathbb{Z}$ with the following properties:</u>

(i) <u>$|A_t(w)| = \lambda_t$ for $1 \le t < \ell$; $|A_\ell(w)| \ge k+1$ for some $0 \le k < \lambda_\ell$.</u>

(ii) <u>For all $1 \le h \le k$, $j_\ell^h(w) > j_1^h(w) > j_2^h(w) > \dots > j_{\ell-1}^h(w)$.</u>

<u>Then there exists $w' = \rho^{A_\ell}_{A_1,\ldots,A_{\ell-1}}(w)$ such that w' has the MDC form $(A_\ell,\ldots,A_1)$ at i with</u>

$$\begin{cases} \underline{j_\ell^{k+1}(w) - n < j_\ell^{k+1}(w') < j_\ell^{k+1}(w)} \\ \underline{j_t^{k+1}(w') \geq j_t^{k+1}(w), \text{ for all } 1 \leq t \leq \ell-1.} \end{cases}$$

<u>Proof:</u> The existence of w' follows from Lemma 5.3.5.

Let $\begin{cases} i_1 = \min\{h \mid k < h \leq \lambda_1, \ j_1^h(w) < j_\ell^{k+1}(w)\} \\ i_u = \min\{h \mid k < h \leq \lambda_u, \ j_u^h(w) < j_{u-1}^{i_{u-1}}(w)\}, \ 1 < u \leq \ell-1 \end{cases}$

Then $\begin{cases} j_\ell^{k+1}(w') = j_{\ell-1}^{i_{\ell-1}}(w) \\ j_t^{k+1}(w') = \max\{j_t^{k+1}(w), \ j_{t-1}^{i_{t-1}}(w)\}, \ 1 \leq t \leq \ell-1. \end{cases}$

So it follows that $j_\ell^{k+1}(w') < j_\ell^{k+1}(w)$ and $j_t^{k+1}(w') \geq j_t^{k+1}(w)$ for $1 \leq t \leq \ell-1$. Suppose $j_\ell^{k+1}(w) - n > j_\ell^{k+1}(w')$. i.e. $j_\ell^{k+1}(w) - n > j_{\ell-1}^{i_{\ell-1}}(w)$. Let

$$S_t = \{e((w), j_{t+1}^u(w)), \ e((w), j_t^v(w)) \mid 1 \leq u < i_{t+1}, \ i_t \leq v \leq \lambda_t\}, \ 1 \leq t < \ell-1$$

$$S_{\ell-1} = \{e((w), j_1^u(w)), \ e((w), j_\ell^{k+1}(w)), \ e((w), j_{\ell-1}^v(w)+n) \mid 1 \leq u < i_1, \ i_{\ell-1} \leq v \leq \lambda_{\ell-1}\}.$$

Then $S = S_1 \cup \ldots \cup S_{\ell-1}$ satisfies $C_n(w, \ell-1)$. But $|S| = \sum_{t=1}^{\ell-1} \lambda_t + 1$. It contradicts $w \in \sigma^{-1}(\lambda)$. So $j_\ell^{k+1}(w) - n < j_\ell^{k+1}(w')$ because $\overline{j_\ell^{k+1}(w)} \neq \overline{j_\ell^{k+1}(w')}$. □

<u>Proposition 7.4</u> <u>For any $w \in H_\lambda$, there exists $y \in N_\lambda$ such that $y \underset{P_L}{\sim} w$.</u>

<u>Proof</u>: Suppose that $w \in H_\lambda$ has a standard MDC form $(A_r,\dots,A_1)$ at i and suppose that k is the largest number with $0 \leq k \leq \lambda_1$ such that for all h with $1 \leq h \leq k$, we have $(j_1^h(w) - n < j_r^h(w) < j_{r-1}^h(w) < \dots < j_1^h(w))^{(om)}$. If $k = \lambda_1$, then $w \in N_\lambda$ and the result is trivial. Now assume $k < \lambda_1$. Then there exists v such that v is the smallest number satisfying $1 < v \leq r$ and $j_1^{k+1}(w) - j_v^{k+1}(w) > n$. By Lemmas 5.4.3 and 3.7, there exists $w' = \rho^{A_r,\dots,A_v}_{A_1,\dots,A_{v-1}}(w)$ in H_λ. By Corollary 5.4.7, we have

$$(j_1^h(w') - n < j_r^h(w') < j_{r-1}^h(w') < \dots < j_1^h(w'))^{(om)} \text{ for all } 1 \leq h \leq k \tag{1}$$

Assume that for some $v \leq m \leq r$, $\lambda_m \geq k+1$ but $\lambda_{m+1} < k+1$ (such an m obviously exists). Let

$$w_1 = \rho^{A_r,\dots,A_{m+1}}_{A_1,\dots,A_{v-1}}(w),\ w_2 = \rho^{A_m}_{A_1,\dots,A_{v-1}}(w_1),\dots,\ w_{m+2-v}$$

$$= \rho^{A_v}_{A_1,\dots,A_{v-1}}(w_{m+1-v}) = w'.$$

Then

$$j_1^h(w_1),\ j_2^h(w_1),\dots,j_{v-1}^h(w_1))^{(om)} = (j_1^h(w),\ j_2^h(w),\dots,j_{v-1}^h(w))^{(om)}$$

$$j_v^h(w_1), j_{v+1}^h(w_1),\dots,j_m^h(w_1))^{(om)} = (j_v^h(w), j_{v+1}^h(w),\dots,j_m^h(w))^{(om)},$$

$$k+1 \leq h \leq \lambda_1$$

$$(j_v^g(w_1)+n, j_{v+1}^g(w_1)+n, \dots, j_m^g(w_1)+n, j_1^g(w_1), j_2^g(w_1), \dots, j_{v-1}^g(w_1),$$
$$j_{m+1}^g(w_1),\ j_{m+2}^g(w_1), \dots, j_r^g(w_1))^{(om)}$$
$$= (j_v^g(w) + n,\ j_{v+1}^g(w)+n, \dots, j_r^g(w)+n, j_1^g(w), j_2^g(w), \dots, j_{v-1}^g(w))^{(om)}, 1<g<k$$

So for $1 < g < k$,

$$(j_v^g(w_1)+n)-n < j_r^g(w_1) < j_{r-1}^g(w_1) < \dots < j_{m+1}^g(w_1) < j_{v-1}^g(w_1) < j_{v-2}^g(w_1) < \dots$$
$$< j_1^g(w_1) < j_m^g(w_1) + n < j_{m-1}^g(w_1)+n < \dots < j_v^g(w_1) + n)^{(om)}$$

By Lemma 7.3, we have

$$\begin{cases} (j_t^{k+1}(w) + n)-n < j_t^{k+1}(w') < j_t^{k+1}(w) + n & \text{for } v < t < m \\ j_{t'}^{k+1}(w') > j_{t'}^{k+1}(w) & \text{for } 1 < t' < v-1 \end{cases}$$

But it is clear that $j_1^{k+1}(w') = j_1^{k+1}(w)$ by our hypothesis.
So this together with Lemma 7.3 implies that

$$\begin{cases} 0 < j_1^{k+1}(w') - j_{t'}^{k+1}(w') < j_1^{k+1}(w) - j_{t'}^{k+1}(w) \text{ for } 1 < t' < v-1 \\ 0 < j_1^{k+1}(w') - j_t^{k+1}(w') < j_1^{k+1}(w) - j_t^{k+1}(w) \text{ for } v < t < m \end{cases}$$

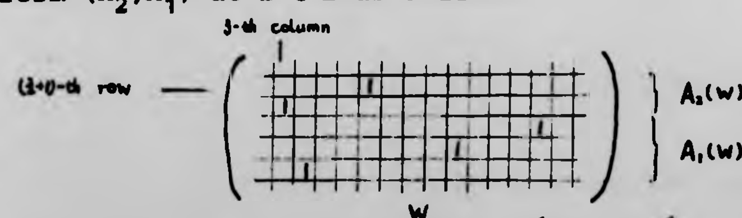
In particular, $0 < j_1^{k+1}(w') - j_v^{k+1}(w') < j_1^{k+1}(w) - j_v^{k+1}(w)$ (2)

Combining (1), (2), we see that whenever $j_1^{k+1}(w) - j_v^{k+1}(w) > n$, we can always find $w' \in H_\lambda$ by a succession of ρ operations such that w' has a standard MDC form $(A_r,\dots,A_1)$ at i' for some $i' \in \mathbb{Z}$ which is normal for the first k layers with $j_1^{k+1}(w') - j_t^{k+1}(w') < n$ for all $1 < t < v$ and $0 < j_1^{k+1}(w') - j_v^{k+1}(w') \lneq j_1^{k+1}(w) - j_v^{k+1}(w)$.

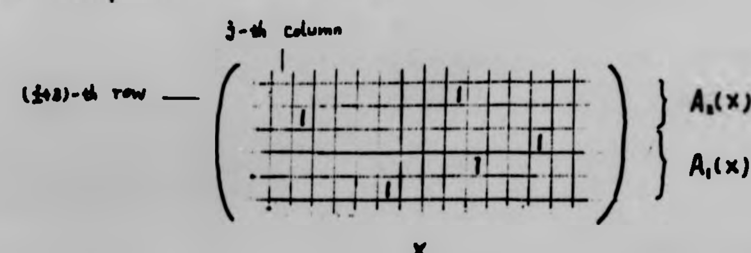
Applying induction on the difference $j_1^{k+1}(w) - j_v^{k+1}(w) > n$, we can finally find $w' \in H_\lambda$ which has all the above properties and also satisfies $0 < j_1^{k+1}(w') - j_v^{k+1}(w') < n$ (note that $\overline{j_1^{k+1}(w')} \neq \overline{j_v^{k+1}(w')}$ always holds). If we denote such an element w' still by w, then the number v for w defined above gets strictly larger. By applying induction on the number $r-v > 0$, we can find $x \in H_\lambda$ with $x \underset{P_L}{\sim} w$ such that x has a standard MDC form which is normal for the first $k+1$ layers. Finally, by applying induction on the number $\lambda_1 - k > 0$, we can get the required $y \in \mathbb{N}_\lambda$. □

We shall finish this chapter by giving an element of H_λ to see how to normalize it into $\mathbb{N}_\lambda$:

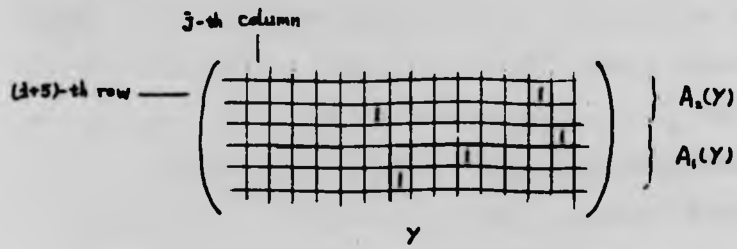
Assume that $n = 5$, $\lambda = \{3 > 2\}$ and $w \in H_\lambda$ has a standard MDC form (A_2, A_1) at $i \in \mathbb{Z}$ as follows:



where j is some integer. Now $j_1^1(w) - j_2^1(w) = 8 > 5$, let $x = \rho^{A_2}_{A_1}(w)$. Then $x \in H_\lambda$ is as follows:



We see that x has a standard MDC form (A_2,A_1) at $i+2$ which is normal for the first layer. Now $j_1^2(x) - j_2^2(x) = 8 > 5$, let $y = \rho_{A_1}^{A_2}(x)$. Then from the following matrix, we see that $y \in H_\lambda$ is a normalized element.



CHAPTER 8 : THE ORBIT SPACE $\tilde{A}_n$ OF THE AFFINE WEYL GROUP A_n

In this chapter, we shall introduce the concept of an unlabelled affine matrix which is essentially a Φ-orbit of A_n, where Φ is an automorphism on A_n defined in §6.2. Let $\tilde{A}_n$ be the set of all Φ-orbits of A_n. Then there exists a natural map $\eta: A_n \to \tilde{A}_n$, most of the results on A_n can be carried over to $\tilde{A}_n$ under this map with slight modification. We shall define an operation of deletion on some special kind of elements $\tilde{w} \in \tilde{A}_n$ which sends $\tilde{w}$ into $\tilde{A}_m$ for some $m < n$. The commutativity between interchanging operations and deletion will enable us to simplify the proofs in the subsequent chapters.

§8.1 DEFINITION OF $\tilde{A}_n$

Let $\tilde{A}_n$ be the set of all matrices $\tilde{w}$ which satisfy the following conditions:

(i) $\tilde{w}$ is an $\infty \times \infty$ matrix with no bounded edge in any direction.

(ii) Each row (resp. column) of $\tilde{w}$ contains a unique non-zero entry and this entry is 1.

(iii) If e, e' are two entries of $\tilde{w}$ with $r(e,e') = c(e,e') \in n\mathbb{Z}$, then e is non-zero if and only if e' is non-zero, where the function $r(\ ,\ )$, $c(\ ,\ )$ are defined in the same way as in A_n.

So $\tilde{w} \in \tilde{A}_n$ can be determined by any of its n consecutive rows (or columns).

For $\tilde{y}, \tilde{y}' \in \tilde{A}_n$, we say $\tilde{y} = \tilde{y}'$ if there exists entries e_o, e'_o (not necessarily non-zero) of $\tilde{y}$, $\tilde{y}'$, respectively, such that for any entries e, e' of $\tilde{y}$, $\tilde{y}'$, respectively, with $r(e_o,e) = r(e_o',e')$ and $c(e_o,e) = c(e_o',e')$, we have that e is non-zero if and only if e' is non-zero.

For any $\tilde{w} \in \tilde{A}_n$, we say an entry e (not necessarily non-zero) of $\tilde{w}$ is diagonal if there exists non-zero entries $e_1,e_2,\ldots,e_n$ of $\tilde{w}$ such that $r(e,e_j) = j$ for any $1 < j < n$ and $\sum_{j=1}^{n} c(e,e_j) = \sum_{j=1}^{n} j$.

<u>Lemma 8.1.1</u> <u>There exists a unique diagonal entry in each row (resp. column) of $\tilde{w}$ for any $\tilde{w} \in \tilde{A}_n$.</u>

<u>Proof</u>: By symmetry, it suffices to show that there exists a unique diagonal entry in each row of $\tilde{w}$. Now let us fix a row of $\tilde{w}$ and take any entry e (not necessarily non-zero) in it. Assume that $e_1,\ldots,e_n$ are non-zero entries of $\tilde{w}$ such that $r(e,e_t) = t$ for $1 < t < n$ and $\sum_{t=1}^{n} c(e,e_t) = m$. Since $c(e,e_t)$'s are incongruent mod n, there exists some $q \in \mathbb{Z}$ such that $m = \sum_{t=1}^{n} t + qn$. Let e' be the entry of $\tilde{w}$ satisfying $r(e,e') = 0$ and $c(e,e') = q$. Then $r(e',e_t) = t$ for $1 < t < n$ and $\sum_{t=1}^{n} c(e',e_t) = \sum_{t=1}^{n} t$. So e' is the diagonal entry in the given row. The uniqueness of e' is obvious. □

By the property that if e, e' are two non-zero entries of $\tilde{w}$ with $r(e,e') \in n\mathbb{Z}$ then $c(e,e') = r(e,e')$, we can easily see that if e, e' are two entries of $\tilde{w}$ (not necessarily non-zero) with $c(e,e') = r(e,e')$ then e and e' are either both diagonal or both not. So the set of all diagonal entries of $\tilde{w}$ forms a diagonal line of $\tilde{w}$ which is uniquely determined.

§8.2 THE MAP $\eta: A_n \to \tilde{A}_n$.

For any $w \in A_n$, let $\eta(w)$ be the matrix which comes from w by forgetting the integers labelling the rows and columns of w. We can check that $\eta(w) \in \tilde{A}_n$. This defines a map $\eta: A_n \to \tilde{A}_n$.

Lemma 8.2.1 η is surjective.

Proof: For any $\tilde{w} \in \tilde{A}_n$, there exists a diagonal entry, say e, of $\tilde{w}$ by Lemma 8.1.1. We give integer labels to all rows and columns of $\tilde{w}$ such that e lies in the (0,0)-position. Then we get an element of A_n. Clearly, the image of this element under η is $\tilde{w}$ and so η is surjective. □

§8.3 THE PARTITION ASSOCIATED WITH AN ELEMENT OF $\tilde{A}_n$

Let ϕ be the automorphism of A_n such that $\phi(s_i) = s_{i+1}$ for any $i \in \mathbb{Z}$. Then the following lemma is clear.

Lemma 8.3.1 For any $\tilde{w} \in \tilde{A}_n$, $\eta^{-1}(\tilde{w})$ is just a ϕ-orbit in A_n. □

Remark 8.3.2 (i) By Lemma 8.3.1, $\tilde{A}_n$ can be regarded as the set of ϕ-orbits of A_n.

(ii) One can show that $|\eta^{-1}(\tilde{w})|\,\big|\,n$, for any $\tilde{w} \in \tilde{A}_n$. Fix $d \in \mathbb{Z}^+$ with $d|n$, let $A_n(d) = \{w \in A_n \,\big|\, |\eta^{-1}(\eta(w))|\,\big|\,d\}$. Then there exists a map $P_d: A_n(d) \to A_d$ by regarding any $w \in A_n(d)$

as an element of A_d. One can check that P_d is bijective.

For $w \in A_n$, assume that $A = \{e_w(i_t, j_t) \mid 1 \le t \le m,$ $i_1 < \dots < i_m,\ j_1 > \dots > j_m\}$ is any descending chain of w, then $A' = \{e_{\phi(w)}(i_t+1,\ j_t+1) \mid 1 \le t \le m\}$ is the corresponding descending chain of $\phi(w)$. By this fact, one can easily see that if $w \in \sigma^{-1}(\lambda)$ then $\phi(w) \in \sigma^{-1}(\lambda)$. So we can define a map $\tilde{\sigma}: \tilde{A}_n \to \Lambda_n$ by $\tilde{\sigma}(\tilde{w}) = \sigma(y)$ for $\tilde{w} \in \tilde{A}_n$ and any $y \in \eta^{-1}(\tilde{w})$. Then we get a commutative diagram as follows:

$$\begin{tikzcd}
A_n \arrow[rr, "\eta"] \arrow[dr, "\sigma"'] & & \tilde{A}_n \arrow[dl, "\tilde{\sigma}"] \\
& \Lambda_n &
\end{tikzcd}$$

§8.4 THE FUNCTIONS $\ell(\tilde{w})$, $\mathcal{L}(\tilde{w})$, $\mathcal{R}(\tilde{w})$ AND STAR OPERATIONS IN $\tilde{A}_n$.

<u>Definition 8.4.1</u> We say that $J = J_1 \cup \dots \cup J_t \in \underline{\Delta}$ is a standard decomposition of J into indecomposable subsets if there exist integers i, m_v, k_v for $1 \le v \le t$ with $0 < m_v \le k_v$ and $\sum_{v=1}^{t} k_v = n$ such that for every $1 \le h \le t$,

$$J_h = \{s_{i+\sum_{v=1}^{h-1} k_v + u} \mid 1 \le u \le m_h\}.$$

Call $(m_1,k_1), (m_2,k_2), \dots, (m_t,k_t)$ the sequence associated with such a standard decomposition of J and call t the number of components of such a standard decomposition of J.

For example, suppose $n = 5$ and $J = \{s_1, s_2, s_4\}$. Then $J = \{s_1, s_2\} \cup \{s_4\}$ is a standard decomposition of J with the associated sequence (2,3), (1,2).

For any $J \in \underline{\Delta}$, we can see that the sequence associated with a standard decomposition of J is uniquely determined by J up to a cyclic permutation, where a sequence $x_1', x_2', \ldots, x_t'$ is said to be a cyclic permutation of a sequence $x_1, x_2, \ldots, x_t$ if there exists an integer h, $1 \leq h \leq t$, such that $(x_1', x_2', \ldots, x_t') = (x_h, x_{h+1}, \ldots, x_t, x_1, x_2, \ldots, x_{h-1})$.

We say $J, J' \in \underline{\Delta}$ are similar, written $J \sim J'$, if there exist standard decompositions of J, J' which have the same associated sequence.

For example, when $n = 5$, $J = \{s_1, s_2\} \cup \{s_4\}$ and $J' = \{s_2, s_3\} \cup \{s_5\}$ have the same associated sequence (2,3), (1,2) and so $J \sim J'$.

This is an equivalence relation. Let $C\ell(\underline{\Delta})$ be the set of all such equivalence classes of $\underline{\Delta}$. Then there exists a natural map from $\underline{\Delta}$ to $C\ell(\underline{\Delta})$ by sending J to $\tilde{J}$, where $\tilde{J} \in C\ell(\underline{\Delta})$ is the class containing J.

For any $\tilde{J}, \tilde{J}' \in C\ell(\underline{\Delta})$, we say $\tilde{J} \subseteq \tilde{J}'$ if there exists $J_0 \in \tilde{J}$ and $J_0' \in \tilde{J}'$ such that $J_0 \subseteq J_0'$. Clearly, this defines a partial order on $C\ell(\underline{\Delta})$.

By Lemma 8.3.1, for any $\tilde{w} \in \tilde{A}_n$, elements y in $\eta^{-1}(\tilde{w})$ have the same length and the sets $\mathcal{L}(y)$ (resp. $\mathcal{R}(y)$) are all similar. So one can define $\ell(\tilde{w}) = \ell(y)$, $\mathcal{L}(\tilde{w}) = \widetilde{\mathcal{L}(y)}$ and $\mathcal{R}(\tilde{w}) = \widetilde{\mathcal{R}(y)}$ for any $y \in \eta^{-1}(\tilde{w})$. If $y \in \mathcal{D}_L(s_t)$ for some $1 \leq t \leq n$, let $y' = {}^*y$ in $\mathcal{D}_L(s_t)$ and let $\tilde{y}' = \eta(y')$. We see that the element $\tilde{y}'$ is independent of the choice of $y \in \eta^{-1}(\tilde{w})$ in the following sense: Suppose $x \in \eta^{-1}(\tilde{w})$ with $x = \phi^h(y)$. Then $x \in \mathcal{D}_L(s_{t+h})$. Let

$x' = {}^*x$ in $D_L(s_{t+h})$ and let $\tilde{x}' = \eta(x')$. Then $\tilde{x}' = \tilde{y}'$. So we can say that $\tilde{y}'$ is obtained from $\tilde{w}$ by a left star operation. Similarly one can define a right star operation on $\tilde{w}$. So one can define the equivalence relations $\underset{L}{\tilde{\sim}}$, $\underset{R}{\tilde{\sim}}$ and $\underset{P}{\sim}$ on $\tilde{A}_n$ in a natural way. One can also define a descending chain, a block, a DC block, an MDC block, a local MDC block, an entry (resp. row, column, row-column) class, etc, of an element $\tilde{w} \in \tilde{A}_n$, and then define ρ, θ operations on $\tilde{w}$ and the sequence $\xi(\tilde{w},\rho)$ (resp. $\xi(\tilde{w},\theta)$) corresponding to ρ(resp. θ) on $\tilde{w}$ in a natural way. For example, we say a non-zero entry set $\{e_1,\ldots,e_t\}$ of $\tilde{w}$ is a descending chain of $\tilde{w}$ if for any $1 < i < j < t$, we have $r(e_i,e_j) \cdot c(e_i,e_j) < 0$. We say a submatrix consisting of any m, $m < n$, consecutive rows of $\tilde{w}$ is a block of $\tilde{w}$. We say $\tilde{w}$ has DC form $(A_\ell,\ldots,A_1)$ if $A_\ell,\ldots,A_1$ are consecutive DC blocks of $\tilde{w}$ downwards with $\sum_{t=1}^{\ell} |A_t| < n$. In that case, e_{tu} (or $e_{tu}(\tilde{w})$) usually denotes the u-th entry of $A_t(\tilde{w})$.

§8.5 <u>INTERCHANGING OPERATIONS ON BLOCKS IN $\tilde{A}_n$</u>.

Since $\tilde{A}_n$ can be regarded as the set of ϕ-orbits of A_n, we can get the analogues for $\tilde{A}_n$ of most of the results on A_n immediately.

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form (A_2,A_1). Set

$$\begin{cases} i_o = 0 \\ i_u = \min\,\{h \mid c(e_{2h},e_{1u}) > 0,\ i_{u-1} < h < |A_2|\},\ 1 < u < |A_1| \end{cases} \tag{8.5.1}$$

(Compare with formula (5.3.2))

Lemma 8.5.2 (Compare with Lemma 5.3.4)

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form (A_2,A_1). Then $\rho_{A_2}^{A_1}(\tilde{w})$ exists $\Longleftrightarrow$ all i_u, $1 \le u \le |A_1|$, exist. When they exist, let $f_{tu}(\tilde{w})$ be the row of $\tilde{w}$ containing $e_{tu}(\tilde{w})$. Then $\tilde{w}' = \rho_{A_2}^{A_1}(\tilde{w})$ is obtained from $\tilde{w}$ by permuting the rows in the block $[A_2(\tilde{w}),\ A_1(\tilde{w})]$ as follows.

$$\begin{cases} f_{1u}(\tilde{w}') \text{ comes from } f_{2,i_u}(\tilde{w}) & \text{for } 1 \le u \le |A_1| \\ f_{2v}(\tilde{w}') \text{ comes from } \begin{cases} f_{2v}(\tilde{w}) \text{ if } v \notin \{i_u \mid 1 \le u \le |A_1|\} \\ f_{1u}(\tilde{w}) \text{ if } v = i_u \text{ for some } 1 \le u \le |A_1| \end{cases} \end{cases} \tag{8.5.3}$$

where $f_{tu}(\tilde{w}')$ is the row of $\tilde{w}'$ containing $e_{tu}(\tilde{w}')$. So $\tilde{w}'$ has local MDC form (A_1,A_2). □

Lemma 8.5.4 (Compare with Lemma 5.3.5)

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form (A_2,A_1). Then $\tilde{w}' = \rho_{A_2}^{A_1}(\tilde{w})$ exists $\Longleftrightarrow$ $A_2(\tilde{w})$ is a longest descending chain in the block $[A_2(\tilde{w}),\ A_1(\tilde{w})]$. □

Lemma 8.5.5 (Compare with Lemma 5.3.7)

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form (A_2,A_1). Then $\tilde{w}' = \theta_{A_1}^{A_2}(\tilde{w})$ exists $\Longleftrightarrow$ $A_1(\tilde{w})$ is a longest descending chain in the block $[A_2(\tilde{w}),\ A_1(\tilde{w})]$. □

In formula (8.5.1), if $\tilde{w}$ has local MDC form (A_2,A_1) which is quasi-normal for the first k layers, then $i_h = h$ for $1 < h < k$. When they exist, let $i_{u+k} = k + i'_{ku}$ for $1 < u < |A_1| - k$, where

$$i'_{ku} = \min\{h \mid c(e_{2,k+h}, e_{1,k+u}) > 0,\ i'_{k,u-1} < h < |A_2| - k\} \qquad (8.5.6)$$

with the convention that $i'_{k0} = 0$. (Compare with formula (5.3.11))

Lemma 8.5.7 (Compare with Corollary 5.3.12)

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form (A_2,A_1) which is quasi-normal for the first k layers. Then

$$\tilde{w}' = \rho_{A_2}^{A_1}(\tilde{w}) \text{ exists} \iff \text{all } i'_{ku},\ 1 < u < |A_1| - k, \text{ exist.} \qquad \square$$

When all i'_{ku}, $1 < u < |A_1| - k$, do exist, we have $|A_2| > |A_1|$.

If $k < |A_1|$, then

$$\begin{cases} \left.\begin{array}{l} f_{1h}(\tilde{w}') \text{ comes from } f_{2h}(\tilde{w}) \\ f_{2h}(\tilde{w}') \text{ comes from } f_{1h}(\tilde{w}) \end{array}\right\} \text{ for } 1 < h < k \\ f_{1,k+u}(\tilde{w}') \text{ comes from } f_{2,i_{k+u}}(\tilde{w}) \quad \text{for } 1 < u < |A_1| - k \\ f_{2,k+v}(\tilde{w}') \text{ comes from } \begin{cases} f_{2,k+v}(\tilde{w}) & \text{for } 1<v<|A_1|-k,\ v\notin\{i'_{ku} \mid 1<u<|A_1|-k\} \\ f_{1,k+u}(\tilde{w}) & \text{for } v = i'_{ku}, \text{ some } 1<u<|A_1|-k \end{cases} \end{cases}$$

If $k > |A_1|$, then

$$\begin{cases} \left.\begin{array}{l} f_{1h}(\tilde{w}') \text{ comes from } f_{2h}(\tilde{w}) \\ f_{2h}(\tilde{w}') \text{ comes from } f_{1h}(\tilde{w}) \end{array}\right\} \text{ for } 1 < h < |A_1| \\ f_{2t}(\tilde{w}') \text{ comes from } f_{2t}(\tilde{w}) \quad \text{for } |A_1| < t < |A_2| \end{cases}$$

We see that the choice of i'_{ku}'s is only dependent on the set of inequalities

$$\{c(e_{2,k+h'}, e_{1,k+h'}) \gtrless 0 \mid 1 < h < |A_2| - k,\ 1 < h' < |A_1| - k\} \quad (8.5.9)$$

in the case that $\tilde{w}$ has local MDC form (A_2, A_1) which is quasi-normal for the first k layers.

Lemma 8.5.10 (Compare with Corollary 5.4.3)

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form $(A_\ell, \ldots, A_1)$. Let $\tilde{w}' = \rho_{A_\ell}^{A_{\ell-1}, \ldots, A_1}(\tilde{w})$. Then $\tilde{w}'$ exists $\Longleftrightarrow A_\ell(\tilde{w})$ is a longest descending chain in the block $[A_\ell(\tilde{w}), \ldots, A_1(\tilde{w})]$. □

Lemma 8.5.11 (Compare with Corollary 5.4.4)

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form $(A_\ell, \ldots, A_1)$. Let $\tilde{w}' = \theta_{A_1}^{A_2, \ldots, A_\ell}(\tilde{w})$. Then $\tilde{w}'$ exists $\Longleftrightarrow A_1(\tilde{w})$ is a longest descending chain in the block $[A_\ell(\tilde{w}), \ldots, A_1(\tilde{w})]$. □

Lemma 8.5.12 (Compare with Lemma 5.4.5)

Assume that $\tilde{w} \in \tilde{A}_n$ has local MDC form $(A_{1\ell_1}, \ldots, A_{11}, A_{2\ell_2}, \ldots, A_{21}, A_{3\ell_3}, \ldots, A_{31}, A_{4\ell_4}, \ldots, A_{41})$ which is quasi-normal (resp. normal) for the first k layers. If there exists $\tilde{w}' = \rho_{A_{21}, \ldots, A_{2\ell_2}}^{A_{3\ell_3}, \ldots, A_{31}}(\tilde{w})$, then $\tilde{w}'$ has local MDC form $(A_{1\ell_1}, \ldots, A_{11}, A_{3\ell_3}, \ldots, A_{31}, A_{2\ell_2}, \ldots, A_{21}, A_{4\ell_4}, \ldots, A_{41})$ which is quasi-normal (resp. normal) for the first k layers. Let $e_{tu}(\tilde{w})$ (resp. $e_{tu}(\tilde{w}')$) be the u-th entry of the t-th block in such a form of $\tilde{w}$ (resp. $\tilde{w}'$). For $1 < u < k$, let $i_1,\ldots,i_t$ with $1 < i_1 < \ldots < i_t < \sum_{\alpha=1}^{4} \ell_\alpha$ (resp. $i_1',\ldots,i_{t'}'$, with $1 < i_1' < \ldots < i_{t'}' < \sum_{\alpha=1}^{4} \ell_\alpha$) be the integer set such that $e_{vu}(\tilde{w})$ exists if and only if $v \in \{i_1,\ldots,i_t\}$ (resp. $e_{vu}(\tilde{w}')$ exists if and only if $v \in \{i_1',\ldots,i_{t'}'\}$). Then $t = t'$ and for any $1 < \alpha, \beta < t$, we have $c(e_{i_\alpha,u}(\tilde{w}), e_{i_\beta,u}(\tilde{w})) = c(e_{i_\alpha',u}(\tilde{w}'), e_{i_\beta',u}(\tilde{w}'))$. □

Lemma 8.5.13 (Compare with Corollary 5.4.7)

Assume that $\tilde{w} \in \tilde{A}_n$ has full MDC form $(A_v,\ldots,A_1, A_\ell,\ldots,A_{v+1})$ which is normal for the first k layers. If there exists $\tilde{w}' = \rho^{A_\ell,\ldots,A_{v+1}}_{A_1,\ldots,A_v}(\tilde{w})$ for some $1 < v < \ell$, then $\tilde{w}'$ has full MDC form $(A_\ell,\ldots,A_1)$ which is normal for the first k layers. □

Lemma 8.5.14 (Compare with Lemma 3.10)

For $\tilde{w} \in \tilde{A}_n$, let $E = \{e_u \mid 1 < u < \ell\} \cup \{e \mid r(e,e_u) = c(e,e_u) \in n\mathbb{Z}$, for some $1 < u < \ell\}$ be a set of entry classes of $\tilde{w}$ such that $0 < r(e_i,e_j) < n$ and $0 < c(e_i,e_j) < n$ for any $1 < i < j < \ell$. Let S be a descending chain of $\tilde{w}$. Then $|E \cap S| < 1$. □

§8.6 TOTALLY ORDERED SETS WITH A DISTANCE FUNCTION

Our purpose of introducing a totally ordered set with a distance function is to establish the commutativity of interchanging operations with deletion.

We say that $\mathcal{R}$ is a totally ordered set with a distance function if $\mathcal{R}$ is a non-empty set with a map $\mathcal{R} \times \mathcal{R} \to \mathbb{Z}$: $(\alpha,\beta) \xrightarrow{<\ ,\ >} <\alpha,\beta>$ satisfying: for any $\alpha,\ \beta,\ \nu \in \mathcal{R}$, we have

$$\begin{cases} \text{(i)} & <\alpha,\beta> = 0 \iff \alpha = \beta \\ \text{(ii)} & <\alpha,\nu> = <\alpha,\beta> + <\beta,\nu> \end{cases} \tag{8.6.1}$$

We can also write $\mathcal{R} = \{\mathcal{R}, <\ ,\ >\}$. Clearly, for any $\alpha,\beta \in \mathcal{R}$, the equation $<\alpha,\beta> = -<\beta,\alpha>$ always holds.

We say $\alpha \leq \beta$ if $<\alpha,\beta> \geq 0$. This is a total order in $\mathcal{R}$. We say a sequence $\xi: \alpha_1,\ldots,\alpha_\ell$ in $\mathcal{R}$ is descending if $<\alpha_i,\alpha_j> < 0$ for any $1 \leq i < j \leq \ell$. Set $|\xi| = \ell$.

Let $\tilde{\mathcal{R}}$ be the set of all descending sequences in $\mathcal{R}$. Let $\tilde{\mathcal{R}}_o^2$ be the set of all ordered pairs (ξ,ζ) with $\xi: \alpha_1,\ldots,\alpha_\ell$ and $\zeta: \beta_1,\ldots,\beta_m$ belonging to $\tilde{\mathcal{R}}$ and $\alpha_t \neq \beta_u$ for any $1 \leq t \leq \ell$, $1 \leq u \leq m$. Let

$$\begin{cases} i_o = 0 \\ i_u = \min\,\{h \mid <\alpha_h,\beta_u> > 0,\ i_{u-1} < h \leq \ell\},\ 1 \leq u \leq m \end{cases} \tag{8.6.2}$$

Clearly, there exists a maximal integer k, $0 \leq k \leq m$, such that $i_o,\ldots,i_k$ exist. We call the sequence $i_o,\ldots,i_k$ the ρ sequence associated with (ξ,ζ). When $k = m$, such a sequence is called a full ρ sequence. Let $\tilde{\mathcal{R}}_1^2 \subseteq \tilde{\mathcal{R}}_o^2$ be the set of all pairs (ξ,ζ) which have a full ρ sequence.

For any $(\xi,\zeta) \in \tilde{\mathcal{R}}_1^2$, let (ξ',ζ') be the ordered pair with $\xi': \alpha_1',\ldots,\alpha_\ell'$ and $\zeta': \beta_1',\ldots,\beta_m'$ such that

$$\begin{cases} \beta'_u = \alpha_{i_u}, \; 1 < u < m \\ \alpha'_v = \begin{cases} \alpha_v \;, \; v \notin \{i_u, \; 1 < u < m\} \\ \beta_u \;, \; v = i_u \text{ for some } 1 < u < m \end{cases} \end{cases} \qquad (8.6.3)$$

Clearly, $(\alpha'_1,\ldots,\alpha'_\ell, \beta'_1,\ldots,\beta'_m)$ is a permutation of $(\alpha_1,\ldots,\alpha_\ell, \beta_1,\ldots,\beta_m)$ and $(\xi',\zeta') \in \tilde{\mathbb{R}}^2_o$. So we have defined a map $\rho:\tilde{\mathbb{R}}^2_1 \to \tilde{\mathbb{R}}^2_o$ by $\rho(\xi,\zeta) = (\xi',\zeta')$. Let ϕ be a map from $\{\alpha_t,\beta_u | 1 < t < \ell, \; 1 < u < m\}$ to $\{1,2,\ldots,\ell+m\}$ such that $\phi(\alpha_t) = t$ and $\phi(\beta_u) = \ell+u$. Set

$$\tilde{\rho}_{\mathbb{R}}(\xi,\zeta) = \begin{pmatrix} 1,\ldots, & \ell & , & \ell+1,\ldots, & \ell+m \\ \phi(\alpha'_1),\ldots,\phi(\alpha'_\ell), & & & \phi(\beta'_1),\ldots,\phi(\beta'_m) \end{pmatrix}$$

Then $\tilde{\rho}_{\mathbb{R}}$ is a map from $\tilde{\mathbb{R}}^2_1$ to $\bigcup_{n\geq 2}^{\infty} S_n$, where S_n is the permutation group on $\{1,2,\ldots,n\}$.

Definition 8.6.4 Assume that $\mathbb{R}$, $\mathbb{R}'$ are two totally ordered sets with distance functions. Let $(\xi,\zeta) \in \tilde{\mathbb{R}}^2_1$, $(\xi',\zeta') \in \tilde{\mathbb{R}}'^2_1$. We say $(\xi,\zeta) \sim (\xi',\zeta')$ if $|\xi| = |\xi'|$, $|\zeta| = |\zeta'|$ and $\tilde{\rho}_{\mathbb{R}}(\xi,\zeta) = \tilde{\rho}_{\mathbb{R}'}(\xi',\zeta')$.

Clearly, $(\xi,\zeta) \sim (\xi',\zeta')$ if and only if their ρ sequences are the same.

Lemma 8.6.5 Assume that $\{\mathbb{R}, <\;,\;>_{\mathbb{R}}\}$, $\{\mathbb{R}', <\;,\;>_{\mathbb{R}'}\}$ are two totally ordered sets with distance functions. Let $(\xi,\zeta) \in \tilde{\mathbb{R}}^2_o$

<u>$(\xi',\zeta') \in \tilde{R}'^{2}_{o}$ with $\xi:\alpha_1,\ldots,\alpha_\ell$; $\zeta:\beta_1,\ldots,\beta_m$; $\xi':\alpha'_1,\ldots,\alpha'_\ell$ and $\zeta':\beta'_1,\ldots,\beta'_m$. If for any t, u with $1 < t < \ell$ and $1 < u < m$, we have $\langle\alpha_t,\beta_u\rangle_R \cdot \langle\alpha'_t,\beta'_u\rangle_{R'} > 0$, then $(\xi,\zeta) \in \tilde{R}^2_1$ if and only if $(\xi'\zeta') \in \tilde{R}'^2_1$. When $(\xi,\zeta) \in \tilde{R}^2_1$ and $(\xi',\zeta') \in \tilde{R}'^2_1$, we have $(\xi,\zeta) \sim (\xi',\zeta')$.</u>

<u>Proof</u>: Let $I:i_o,\ldots,i_k$ and $I':i'_o,\ldots,i'_{k'}$ be the ρ sequences of (ξ,ζ) and (ξ',ζ'), respectively. Then to reach our goal, it suffices to show that $k = k'$ and $(i_o,\ldots,i_k) = (i'_o,\ldots,i'_{k'})$. Clearly, $i_o = i'_o = 0$. In general, assume $i'_h = i_h$ for some u, $u < m$ and all h, $0 < h < u$. We have $\langle\alpha_h,\beta_{u+1}\rangle_R \cdot \langle\alpha'_h,\beta'_{u+1}\rangle_{R'} > 0$ for all h, $i_u < h < \ell$ and $\langle\alpha_t,\beta_{u+1}\rangle_R < \langle\alpha_{t+1},\beta_{u+1}\rangle_R$, $\langle\alpha'_t,\beta'_{u+1}\rangle_{R'} < \langle\alpha'_{t+1},\beta'_{u+1}\rangle_{R'}$ for any t, $i_u < t < \ell$. By formula 8.6.2, this implies that either i_{u+1}, i'_{u+1} both do not exist or $i_{u+1} = i'_{u+1}$. By induction on u, we have $k' = k$ and $(i_o,\ldots,i_k) = (i'_o,\ldots,i'_{k'})$. □

<u>Example 8.6.6</u> For any block, say A, of $w \in A_n$ with $|A| < n$, let ξ_A be the set of integers labelling the columns of w which contain some entry of A. For any j, $j' \in \xi_A$, we define a map $d:(j,j') \to j'-j$. Then $\{\xi_A, d(\ ,\ )\}$ is a totally ordered set with a distance function. We have $\tilde{\xi}_A = \{\xi:j_1,j_2,\ldots,j_\alpha | j_1 > j_2 > \ldots > j_\alpha,\ j_i \in \xi_A\}$. $(\tilde{\xi}_A)^2_o$ is the set of all pairs (ξ,ζ) of $\tilde{\xi}_A$ such that all terms of ξ,ζ are distinct. $(\tilde{\xi}_A)^2_1$ is the set of all pairs $(\xi_2,\xi_1) \in (\tilde{\xi}_A)^2_o$ with $\xi_t:j_{t1},j_{t2},\ldots,j_{tm_t}$, $t = 1,2$

such that for any $1 < u < m_2$, $|\{h | j_{2u} - j_{1h} > 0,\ 1 < h < m_1\}| < m_2 - u$. In particular, if A_2, A_1 are two consecutive DC subblocks of A with A_2 lying above A_1, let $\xi_t : j_t^1(w), \ldots, j_t^{m_t}(w)$, $t = 1,2$, where $|A_t| = m_t$ and $e((w), j_t^u(w))$ is the u-th entry of A_t. Then

$$\rho_{A_2}^{A_1}(w) \text{ exists} \longleftrightarrow (\xi_2, \xi_1) \in (\tilde{\xi}_A)_1^2$$

by definition of ρ operation on w. If $w' = \rho_{A_2}^{A_1}(w)$ does exist, let $\xi_t' : j_t'(w'), j_t^2(w'), \ldots, j_t^{m_t}(w')$, $t = 1,2$. Then it is clear that $(\xi_2', \xi_1') \in (\tilde{\xi}_A)_o^2$ satisfies formula 8.6.3 by replacing (ξ, ζ) and (ξ', ζ') by (ξ_2, ξ_1) and (ξ_2', ξ_1'), respectively, i.e. $\rho(\xi_2, \xi_1) = (\xi_2', \xi_1')$.

<u>Example 8.6.7</u> For any block, say A, of $\tilde{w} \in \tilde{A}_n$ with $|A| < n$, let ζ_A be the set of entries of A with the map $(e,e') \xrightarrow{c(\ ,\ )} c(e,e')$ Then $\{\zeta_A, c(\ ,\ )\}$ is a totally ordered set with a distance function. We have $\tilde{\zeta}_A = \{\zeta : e_{i_1}, \ldots, e_{i_\alpha} | c(e_{i_t}, e_{i_u}) < 0 \text{ for any } t < u\}$. $(\tilde{\xi}_A)_o^2$ is the set of all pairs (ξ, ζ) of $\tilde{\xi}_A$ such that all terms of ξ, ζ are distinct. $(\tilde{\xi}_A)_1^2$ is the set of all pairs $(\zeta_2, \zeta_1) \in (\tilde{\zeta}_A)_o^2$ with $\zeta_t : e_{t1}, \ldots, e_{tm_t}$, $t = 1,2$, such that for any $1 < u < m_2$, $|\{h | c(e_{1h}, e_{2u}) > 0,\ 1 < h < m_1\}| < m_2 - u$. In particular, if A_2, A_1 are two consecutive DC subblocks of A with A_2 lying above A_1, let (ζ_2, ζ_1) be as above with $|A_t| = m_t$ and e_{tu} the u-th entry of A_t. Then

$$\rho_{A_2}^{A_1}(\tilde{w}) \text{ exists} \longleftrightarrow (\zeta_2, \zeta_1) \in (\tilde{\zeta}_A)_1^2$$

by definition of ρ operation on $\tilde{w}$. If $\tilde{w}' = \rho^{A_1}_{A_2}(\tilde{w})$ does exist, assume that $f_{tu}(\tilde{w}')$ is the row of $\tilde{w}'$ containing the u-th entry of $A_t(\tilde{w}')$ and assume that $f_{tu}(\tilde{w}')$ comes from the row $f_{tu}(\tilde{w})$ of $\tilde{w}$ under the operation $\rho^{A_1}_{A_2}$. Let e'_{tu} be the entry of $f_{tu}(\tilde{w})$. Then $e'_{tu} \in \zeta_A$. Let $\zeta_t : e'_{t1}, \ldots, e'_{tm_t}$, $t = 1,2$. Then we see that $\rho(\zeta_2, \zeta_1) = (\zeta'_2, \zeta'_1) \in (\tilde{\zeta}_A)^2_o$.

§8.7 DELETION OPERATIONS IN $\tilde{A}_n$

For a given set E of entry classes of $\tilde{w} \in \tilde{A}_n$, we define the corresponding set ζ_E of rc-classes of $\tilde{w}$ to be the set of all rows and columns of $\tilde{w}$ which contain some entry of E. The size of ζ_E is defined to be the size of E, and is written $|\zeta_E|$. Every rc-class of $\tilde{w}$ is defined in this way.

Let ζ be a set of rc-classes of $\tilde{w} \in \tilde{A}_n$ with $0 < |\zeta| = m < n$ and let $\tilde{v}$ be the matrix obtained from $\tilde{w}$ by deleting all rows and columns of ζ. Then $\tilde{v} \in \tilde{A}_{n-m}$. In that case, let e, e' be any two entries of $\tilde{w}$ not lying in ζ. Then the corresponding entries e_o, e'_o of e, e' in $\tilde{v}$ satisfy

$$\begin{cases} \underline{r_{\tilde{v}}(e_o,e'_o) = r_{\tilde{w}}(e,e') - a_{\tilde{w},\zeta}(e,e')} \\ \underline{c_{\tilde{v}}(e_o,e'_o) = c_{\tilde{w}}(e,e') - b_{\tilde{w},\zeta}(e,e')} \end{cases} \qquad (8.7.1)$$

where $a_{\tilde{w},\zeta}(e,e')$ (resp. $b_{\tilde{w},\zeta}(e,e')$) is the number of rows (resp. columns) of ζ lying between e and e'.

Let $\ell = n-m$. Then we have the following results.

Lemma 8.7.2. (i) $r_{\tilde{v}}(e_o,e_o') > 0 \Longleftrightarrow r_{\tilde{w}}(e,e') > 0$

$r_{\tilde{v}}(e_o,e_o') < 0 \Longleftrightarrow r_{\tilde{w}}(e,e') \leq 0$

(ii) For any $q \in \mathbb{Z}$, $r_{\tilde{v}}(e_o,e_o') = q\ell \Longleftrightarrow r_{\tilde{w}}(e,e') = qn$

(iii) $0 < r_{\tilde{v}}(e_o,e_o') < \ell \Longleftrightarrow 0 < r_{\tilde{w}}(e,e') < n$

(iv) Let $r_{\tilde{v}}(e_o,e_o') = q'\ell + r'$, $r_{\tilde{w}}(e,e') = qn + r$ with $q, q', r, r' \in \mathbb{Z}$, $0 < r' < \ell$ and $0 < r < n$. Then $q = q'$.

In (i) - (iv), by replacing $r_{\tilde{v}}(\ ,\ )$, $r_{\tilde{w}}(\ ,\ )$ by $c_{\tilde{v}}(\ ,\ )$, $c_{\tilde{w}}(\ ,\ )$, we have the corresponding results.

Proof: (i) - (iii) are obvious. For (iv), let e" be the entry of $\tilde{w}$ such that $r_{\tilde{w}}(e,e") = qn$. Then e" does not lie in ζ. So there exists the corresponding entry $e_o"$ of e" in $\tilde{v}$. By (ii), we have $r_{\tilde{v}}(e_o,e_o") = q\ell$. Since $r_{\tilde{w}}(e",e') = r_{\tilde{w}}(e",e) + r_{\tilde{w}}(e,e') = r_{\tilde{w}}(e,e') - r_{\tilde{w}}(e,e") = (qn+r) - qn = r$, we have $0 < r_{\tilde{w}}(e",e') < n$. Hence by (iii), we have $0 < r_{\tilde{v}}(e_o",e_o') < \ell$. So $r_{\tilde{v}}(e_o,e_o') = r_{\tilde{v}}(e_o,e_o") + r_{\tilde{v}}(e_o",e_o') = q\ell + r_{\tilde{v}}(e_o",e_o')$. By uniqueness of such an expression, it follows that $q = q'$. □

When $\tilde{v}$ is obtained from $\tilde{w}$ deleting ζ, we can also say that a block of $\tilde{v}$ is obtained from a block, say A, of $\tilde{w}$ by deleting all rows and columns in ζ. We usually denote such a block of $\tilde{v}$ also by A. For the sake of distinction, we write $A(\tilde{w})$, $A(\tilde{v})$, respectively.

§8.8 COMMUTATIVITY OF INTERCHANGING OPERATIONS WITH DELETION

Assume that $\tilde{w} \in \tilde{A}_n$ has a local MDC form (A_2,A_1) which is quasi-normal for the first k layers. Let $\tilde{v} \in \tilde{A}_{n-2k}$ be obtained from $\tilde{w}$ by deleting ζ, where ζ is the set of rc-classes of $\tilde{w}$ containing all entries $e_{tu}(\tilde{w})$, $1 < t < 2$, $1 < u < k$ with $|\zeta| = 2k$.

<u>Lemma 8.8.1</u> <u>$\tilde{w}' = \rho^{A_1}_{A_2}(\tilde{w})$ exists $\Longleftrightarrow$ $\tilde{v}' = \rho^{A_1}_{A_2}(\tilde{v})$ exists.</u>

<u>When they do exist, let ζ' be the set of rc-classes of $\tilde{w}'$ containing all $e_{tu}(\tilde{w}')$, $t = 1,2$, $1 < u < k$ with $|\zeta'| = 2k$.</u>

<u>Let $\tilde{v}" \in \tilde{A}_{n-2k}$ be obtained from $\tilde{w}'$ by deleting ζ'. Then $\tilde{v}" = \tilde{v}'$.</u>

<u>Proof</u>: Clearly $\mathcal{R} = \{\{e_{t,k+u}(\tilde{w}) \,|^{t=1,2}_{1<u<m_t-k}\}, c_{\tilde{w}}(\ ,\ )\}$,

$\mathcal{R}' = \{\{e_{tu}(\tilde{v}) \,|^{t=1,2}_{1<u<m_t-k}\}, c_{\tilde{v}}(\ ,\ )\}$ are two totally ordered sets with distance functions, where $m_t = |A_t(\tilde{w})|$, $t = 1,2$. Let

$$\left\{\begin{array}{l} \zeta_t(\tilde{w}) : e_{t,k+1}(\tilde{w}), \ldots, e_{t,m_t}(\tilde{w}) \\ \zeta_t(\tilde{v}) : e_{t1}(\tilde{v}), \ldots, e_{t,m_t-k}(\tilde{v}) \end{array}\right. \qquad \text{for } t = 1,2.$$

Then by Lemma 8.5.7, we have

$$\tilde{w}' = \rho^{A_1}_{A_2}(\tilde{w}) \text{ exists} \longleftrightarrow (\zeta_2(\tilde{w}), \zeta_1(\tilde{w})) \in \tilde{\mathcal{R}}^2_1$$

$$\tilde{v}' = \rho^{A_1}_{A_2}(\tilde{v}) \text{ exists} \longleftrightarrow (\zeta_2(\tilde{v}), \zeta_1(\tilde{v})) \in \tilde{\mathcal{R}}'^2_1.$$

Since for any u, u' with $1 < u < m_2-k$, $1 < u' < m_1-k$, we have $c_{\tilde{w}}(e_{2,k+u}(\tilde{w}), e_{1,k+u'}(\tilde{w})) \cdot c_{\tilde{v}}(e_{2,u}(\tilde{v}), e_{1,u'}(\tilde{v})) > 0$, we see by Lemma 8.6.5 that $(\zeta_2(\tilde{w}), \zeta_1(\tilde{w})) \in \tilde{\mathcal{R}}^2_1$ if and only if $(\zeta_2(\tilde{v}), \zeta_1(\tilde{v})) \in \tilde{\mathcal{R}}'^2_1$. It follows that $\tilde{w}'$ exists if and only if $\tilde{v}'$ exists. When they both exist, also by Lemma 8.6.5, we have

$(\zeta_2(\tilde{w}), \zeta_1(\tilde{w})) \sim (\zeta_2(\tilde{v}), \zeta_1(\tilde{v}))$. So by Formula 8.5.3, we get $\tilde{v}'' = \tilde{v}'$. □

Assume that $\tilde{w}$ has an MDC form $(A_\ell,\ldots,A_1)$ which is normal for the first k layers with $k \le \min\{|A_t| \mid 1 \le t \le \ell\}$. Let $\zeta(\tilde{w})$ be the set of rc-classes containing all e_{tu}, $1 \le t \le \ell$, $1 \le u \le k$ with $|\zeta(\tilde{w})| = \ell k$. Let $\tilde{v} \in \tilde{A}_{n-\ell k}$ be obtained from $\tilde{w}$ by deleting $\zeta(\tilde{w})$. Then $\tilde{v}$ has the corresponding DC form $(A_\ell,\ldots,A_1)$ with $|A_t(\tilde{v})| = |A_t(\tilde{w})|-k$ (where we admit some $|A_t(\tilde{v})| = 0$) for $1 \le t \le \ell$ and the u-th entry of $A_t(\tilde{v})$ comes from the (k+u)-th entry of $A_t(\tilde{w})$.

Lemma 8.8.1 implies the following corollary.

<u>Corollary 8.8.2</u> <u>Let $\tilde{w}$, $\tilde{v}$ be as above. Then for any $1 \le j \le \ell$, $\tilde{w}' = \rho^{A_j}_{A_{j+1}}(\tilde{w})$ exists $\iff \tilde{v}' = \rho^{A_j}_{A_{j+1}}(\tilde{v})$ exists. When they both exist, let $\zeta(\tilde{w}')$ be the set of rc-classes of $\tilde{w}'$ containing all $e_{tu}(\tilde{w}')$, $1 \le t \le \ell$, $1 \le u \le k$ with $|\zeta(\tilde{w}')| = \ell k$. Let $\tilde{v}'' \in \tilde{A}_{n-\ell k}$ be obtained from $\tilde{w}'$ by deleting $\zeta(\tilde{w}')$. Then $\tilde{v}'' = \tilde{v}'$.</u>

<u>Proof</u>: Let $\zeta_{j,j+1}(\tilde{w})$ be the set of rc-classes of $\tilde{w}$ containing all $e_{tu}(\tilde{w})$, $t = j$, $j+1$, $1 \le u \le k$ with $|\zeta_{j,j+1}(\tilde{w})| = 2k$. Let $\tilde{u} \in \tilde{A}_{n-2k}$ be obtained from $\tilde{w}$ by deleting $\zeta_{j,j+1}(\tilde{w})$. Then $\tilde{v} \in \tilde{A}_{n-\ell k}$ is obtained from $\tilde{u}$ by deleting $\zeta_{\hat{j},\widehat{j+1}}(\tilde{u})$, where $\zeta_{\hat{j},\widehat{j+1}}(\tilde{u})$ is the set of rc-classes of $\tilde{u}$ containing all $e_{tu}(\tilde{u})$, $1 \le t \le \ell$, $t \ne j$, $j+1$, $1 \le u \le k$, with $|\zeta_{\hat{j},\widehat{j+1}}(\tilde{u})| = (\ell-2)k$. By Lemma 8.8.1, $\tilde{y} = \rho^{A_j}_{A_{j+1}}(\tilde{w})$ exists $\iff \tilde{u}_1 = \rho^{A_j}_{A_{j+1}}(\tilde{u})$ exists. When they both exist, let $\zeta_{j,j+1}(\tilde{y})$ be the set of rc-classes

of $\tilde{y}$ containing all $e_{tu}(\tilde{y})$, $t = j, j+1$, $1 < u < k$ with $|\zeta_{j,j+1}(\tilde{y})|=2k$. Then also by Lemma 8.8.1, $\tilde{u}_1$ is obtained from $\tilde{y}$ by deleting $\zeta_{j,j+1}(\tilde{y})$, i.e. the diagram

$$\begin{tikzcd}
\tilde{w} \arrow[r, "\text{deleting } \zeta_{j,j+1}(\tilde{w})"] \arrow[d, "\rho^{A_j}_{A_{j+1}}"'] & \tilde{u} \arrow[d, "\rho^{A_j}_{A_{j+1}}"] \\
\tilde{y} \arrow[r, "\text{deleting } \zeta_{j,j+1}(\tilde{y})"] & \tilde{u}_1
\end{tikzcd}$$

commutes. But by the same argument as that in the proof of Lemma 8.8.1, we can show that

$$\tilde{u}_1 = \rho^{A_j}_{A_{j+1}}(\tilde{u}) \text{ exists} \Longleftrightarrow \tilde{x} = \rho^{A_j}_{A_{j+1}}(\tilde{v}) \text{ exists.}$$

When they both exist, the diagram

$$\begin{tikzcd}
\tilde{u} \arrow[r, "\text{deleting } \zeta_{\hat{j},\widehat{j+1}}(\tilde{u})"] \arrow[d, "\rho^{A_j}_{A_{j+1}}"'] & \tilde{v} \arrow[d, "\rho^{A_j}_{A_{j+1}}"] \\
\tilde{u}_1 \arrow[r, "\text{deleting } \zeta_{\hat{j},\widehat{j+1}}(\tilde{u}_1)"] & \tilde{x}
\end{tikzcd}$$

commutes. Combining the above two results, we reach our goal. □

§8.9 COMMUTATIVITY OF INTERCHANGING OPERATIONS WITH THE MAP η.

Assume that $w \in A_n$ has a DC form (A_2, A_1) at $i \in \mathbb{Z}$. We assume that the corresponding DC form of $\tilde{w} = \eta(w)$ is also

(A_2, A_1). Then by definition of ρ, θ operations on both w and $\tilde{w}$, the following lemma is immediate.

Lemma 8.9.1 Assume that $w \in \Lambda_n$ has a DC form (A_2, A_1) at $i \in \mathbb{Z}$. Then

$$w' = \rho_{A_2}^{A_1}(w) \text{ exists} \longleftrightarrow \eta(w)' = \rho_{A_2}^{A_1}(\eta(w)) \text{ exists.}$$

Also, $w'' = \theta_{A_1}^{A_2}(w)$ exists $\longleftrightarrow \eta(w)'' = \theta_{A_1}^{A_2}(\eta(w))$ exists.

When they all exist , the diagrams

$$\begin{tikzcd}
w \arrow[r, "\eta"] \arrow[d, "\rho_{A_2}^{A_1}"'] & \eta(w) \arrow[d, "\rho_{A_2}^{A_1}"] \\
w' \arrow[r, "\eta"] & \eta(w')
\end{tikzcd}$$

$$\begin{tikzcd}
w \arrow[r, "\eta"] \arrow[d, "\theta_{A_1}^{A_2}"'] & \eta(w) \arrow[d, "\theta_{A_1}^{A_2}"] \\
w'' \arrow[r, "\eta"] & \eta(w'')
\end{tikzcd}$$

commute. □

CHAPTER 9 : THE SEQUENCE $\xi(w,k)$ BEGINNING WITH AN ELEMENT OF N_λ.

In the present chapter, the partition $\lambda = \{\lambda_1 > \dots > \lambda_r\} \in \Lambda_n$ is fixed. $\mu = \{\mu_1 > \dots > \mu_m\}$ is the dual partition of λ. To avoid the trivial case, we assume $\lambda \neq \{1 > \dots > 1\}$.

In chapter 10, we shall introduce a raising operation on an element w of N_λ and show that if w' is obtained from w by a raising operation and $w' \in N_\lambda$ then w' lies in the same left cell as w. In order to do this, we shall make use of a certain sequence $\xi(w,k)$ which is discussed in this chapter.

§9.1 A DESCRIPTION OF N_λ.

Lemma 9.1.1 Assume that $w \in A_n$ has a full MDC form $(A_r,\dots,A_1)$ at $i \in \mathbb{Z}$ which is normal with $|A_t(w)| = \lambda_t$, $1 < t < r$. Then $w \in \sigma^{-1}(\lambda)$ and so $w \in N_\lambda$.

Proof: Assume that $w \in \sigma^{-1}(\lambda')$ for some $\lambda' = \{\lambda'_1 > \dots > \lambda'_r\} \in \Lambda_n$. It suffices to show $\lambda' = \lambda$. We claim $\lambda' > \lambda$. For, let $S_t = \{e((w), j_t^u(w)) \mid 1 < u < \lambda_t\}$, $1 < t < r$. Then $S = S_1 \cup \dots \cup S_r$ satisfies $C_n(w,r)$ with $|S_1 \cup \dots \cup S_t| = \sum_{h=1}^{t} \lambda_h$ for any $1 < t < r$.

On the other hand, if for some $1 < t' < r'$, $S' = S'_1 \cup \dots \cup S'_{t'}$ satisfies $C_n(w,t')$, let $\bar{e}_u$ be the u-th layer of w, $1 < u < \lambda_1$. Then by Lemma 3.10, $|\bar{e}_u \cap S'_v| < 1$. So $|(\bigcup_{v=1}^{t'} S'_v) \cap \bar{e}_u| < \min\{\mu_u, t'\}$ for $1 < u < \lambda_1$. This implies $|S'| < \sum_{h=1}^{t'} \lambda_h$. So $\lambda' = \lambda$ and $w \in \sigma^{-1}(\lambda)$. The latter part of this lemma is immediate. □

<u>Remark 9.1.2</u> The above lemma gives a sufficient condition for an element of $\mathbb{A}_n$ to belong to $\mathbb{H}_\lambda$. Obviously, it is also a necessary condition.

§9.2 <u>A SEQUENCE $\xi(w,r)$ BEGINNING WITH AN ELEMENT OF H_λ.</u>

We begin with the case $k = r$ since this is considerably easier than the general case and the general case will make use of this special case.

Now assume that $\lambda_r > 1$.

<u>Lemma 9.2.1</u> <u>Assume that $w \in \sigma^{-1}(\lambda)$ has a full DC form $(A_{r+1},\dots,A_1)$ at $i \in \mathbb{Z}$ with $|A_t(w)| = \lambda_t$, $1 \le t \le r$, $|A_r(w)| = \lambda_r - 1$ and $|A_{r+1}(w)| = 1$. Suppose that for some $1 \le m \le \lambda_r$, $j_r^m(w) < j_{r+1}^1(w) < j_r^{m-1}(w)$ with the convention that $j_r^0(w) = \infty$, $j_r^{\lambda_r}(w) = -\infty$. Then there exists $w' = \rho^{A_{r+1}}_{A_1,\dots,A_{r-1}}(w)$ having the full DC form $(A_r, A_{r+1}, A_{r-1},\dots,A_1)$ at $i+1$ with $j_r^m(w') < j_{r+1}^1(w) < j_{r+1}^1(w') < j_r^{m-1}(w')$.</u>

<u>Proof</u>: The existence of w' follows by Lemma 5.3.5 and Lemma 3.7. Let

$$\begin{cases} i_1 = \min\{h \mid 1 \le h \le \lambda_1,\ j_1^h(w) < j_{r+1}^1(w) + n\} \\ i_u = \min\{h \mid 1 \le h \le \lambda_u,\ j_u^h(w) < j_{u-1}^{i_{u-1}}(w)\}, \quad 1 < u \le r-1 \end{cases}$$

Then by repeatedly applying Lemma 5.3.4, all $i_1,\dots,i_{r-1}$ exist and $j_{r+1}^1(w') = j_{r-1}^{i_{r-1}}(w)$. If $j_{r-1}^{i_{r-1}}(w) > j_r^{m-1}(w)$, then

$j_r^{m-1}(w) + n > j_{r+1}^1(w) + n > j_1^{i_1}(w) > j_2^{i_2}(w) > \dots > j_{r-1}^{i_{r-1}}(w) > j_r^{m-1}(w)$.

By Lemma 3.10, we need at least $(r+1)$ descending chains of w to contain an entry set $\{e((w),\bar{j}_v) \mid 1 < v < r+1\}$ with $\{\bar{j}_v \mid 1 < v < r+1\} = \{\overline{j_v^{i_v}(w)}, 1 < v < r-1; \overline{j_{r+1}^{1}(w)}, \overline{j_r^{m-1}(w)}\}$. This contradicts $w \in \sigma^{-1}(\lambda)$. So $j_{r-1}^{i_{r-1}}(w) < j_r^{m-1}(w)$ (since $\overline{j_{r-1}^{i_{r-1}}(w)} \neq \overline{j_r^{m-1}(w)}$.) i.e. $j_{r+1}^{1}(w') < j_r^{m-1}(w')$.

On the other hand, if $j_{r-1}^{i_{r-1}}(w) < j_{r+1}^{1}(w)$, let

$$S_1 = \{e((w),j_1^u(w)) \mid 1 < u < i_1\} \cup \{e((w),j_{r+1}^{1}(w)+n)\} \cup \{e((w),j_{r+1}^{v}(w)+n) \mid i_{r-1} < v < \lambda_{r-1}\}$$

$$S_h = \{e((w),j_h^u(w)) \mid 1 < u < i_h\} \cup \{e((w),j_{h-1}^{v}(w)) \mid i_{h-1} < v < \lambda_{h-1}\}, \quad 1 < h < r-1.$$

Then $S = S_1 \cup \ldots \cup S_{r-1}$ satisfies $C_n(w,r-1)$ with $|S| = \sum_{h=1}^{r-1} \lambda_h + 1$. This also contradicts $w \in \sigma^{-1}(\lambda)$. So $j_{r-1}^{i_{r-1}}(w) > j_{r+1}^{1}(w)$ (also since $\overline{j_{r-1}^{i_{r-1}}(w)} \neq \overline{j_{r+1}^{1}(w)}$). i.e. $j_{r+1}^{1}(w') > j_{r+1}^{1}(w)$. Since $j_{r+1}^{1}(w) > j_r^m(w) = j_r^m(w')$, it follows that

$j_r^m(w') < j_{r+1}^{1}(w) < j_{r+1}^{1}(w') < j_r^{m-1}(w')$. □

For $1 < m < \lambda_r$, let $H_{\lambda,r}^m$ be the set of all elements $w \in \sigma^{-1}(\lambda)$ which have a full DC form $(A_{r+1},\ldots,A_1)$ at i for some $i \in \mathbb{Z}$ with $|A_t(w)| = \lambda_t$, $1 < t < r$, $|A_r(w)| = \lambda_r - 1$, $|A_{r+1}(w)| = 1$ and $j_r^m(w) < j_{r+1}^{1}(w) < j_r^{m-1}(w)$. Then it is clear that $H_\lambda = H_{\lambda,r}^1$.

__Lemma 9.2.2__ __For any $w \in H_{\lambda,r}^m$, $1 < m < \lambda_r$, there exists $x \in H_{\lambda,r}^{m+1}$ such that $x = \rho_{A_1,\ldots,A_r}^{A_{r+1}}(w)$.__

<u>Proof</u>: As in Lemma 9.2.1, there exists $w' = \rho^{A_{r+1}}_{A_1,\ldots,A_{r-1}}(w)$ such that $j^m_r(w') < j^1_{r+1}(w') < j^{m-1}_r(w')$. So there exists $x = \rho^{A_{r+1}}_{A_r}(w')$. Obviously, $x \in H^{m+1}_{\lambda,r}$, as required. □

By Lemma 9.2.2, for any $w \in H_\lambda$ which has a standard MDC form $(A_r,\ldots,A_1)$ at $i \in \mathbb{Z}$, let A^0_r (resp. A^1_r) be the block consisting of the first row (resp. the last λ_r-1 rows) of A_r. Then there exists a sequence of elements $w_0 = w, w_1,\ldots,w_{\lambda_r}$ such that for every $1 \leq j < \lambda_r$, $w_j = \rho^{A^0_r}_{A_1,\ldots,A_{r-1},A^1_r}(w_{j-1}) \in H^{j+1}_{\lambda,r}$, $w_{\lambda_r} = \rho^{A^0_r}_{A_1,\ldots,A_{r-1}}(w_{\lambda_r-1}) \in H_\lambda$.

Let $\xi(w,r)$ be the sequence obtained by linking all the sequences $\xi(w_{j-1},\rho^{A^0_r}_{A_1,\ldots,A_{r-1},A^1_r})$, $1 \leq j < \lambda_r$, $\xi(w_{\lambda_r-1},\rho^{A^0_r}_{A_1,\ldots,A_{r-1}})$ together. (These sequences have been defined in §5.4). We can see that when $\lambda_1 \neq \lambda_r$, such a sequence exists and is unique, but when $\lambda_1 = \lambda_r$ and $r > 1$, there may exist r different such sequences beginning with w because we can name any MDC block of w by A_r. But in either case, the length of such a sequence is independent of the choice of w in H_λ and of the choice of the standard MDC forms.

The above results still hold even when $\lambda_r = 1$ (the trivial case). In that case, the sequence $\xi(w,r)$ is just $\xi(w,\rho^{A^0_r}_{A_1,\ldots,A_{r-1}})$. So we need make no restriction on λ_r.

§9.3 THE DELETION MAP $d(\lambda,m)$

In the remainder of this chapter, we shall make use of the deletion operation introduced in Chapter 8 to relate the construction of the sequence $\xi(w,k)$ to the special case $k = r$ which has just been considered. In this section, we shall define a deletion operator $d(\lambda,m)$ on a special subset $\tilde{F}(\lambda,m)$ of $\tilde{\sigma}_n^{-1}(\lambda)$ and give some result for this operator, where $\tilde{\sigma}_n$ is just $\tilde{\sigma}$ on $\tilde{\Lambda}_n$.

We define $\tilde{F}(\lambda,m)$ to be the set of all elements $\tilde{w} \in \tilde{\sigma}_n^{-1}(\lambda)$ which have the full MDC form $(A_r,\ldots,A_1)$ which is normal for the first m layers with $|A_h(\tilde{w})| > m$ for all $1 < h < r$. Let $\tau = \{\lambda_1 - m > \ldots > \lambda_r - m\} \in \Lambda_{n-rm}$. We define a map $d(\lambda,m):\tilde{F}(\lambda,m) \to \tilde{\Lambda}_{n-rm}$ by deleting $\zeta(\tilde{w},m)$ from $\tilde{w} \in \tilde{F}(\lambda,m)$, where $\zeta(\tilde{w},m)$ is the set of rc-classes of $\tilde{w}$ containing the first m entries of all MDC blocks of $\tilde{w}$ with $|\zeta(\tilde{w},m)| = rm$.

Lemma 9.3.1 Im $d(\lambda,m) \subseteq \tilde{\sigma}_{n-rm}^{-1}(\tau)$.

To show this lemma, we need the following result.

Lemma 9.3.2 Assume that $w \in \sigma^{-1}(\lambda)$ has a full DC form $(A_\ell,\ldots,A_1)$ at i. Then for any t, $1 < t < r$, there exists $S = S_1 \cup \ldots \cup S_t$ satisfying $C_n(w,t)$ and also satisfying the following conditions.

(i) Let $|S_h| = m_h$, $1 < h < t$. Then $\sum_{h=1}^{t} m_h = \sum_{h=1}^{t} \lambda_h$.

(ii) Let $S_h = \{e(i_{hk},j_{hk}) \mid 1 < k < m_h, i_{h_1}<\ldots<i_{hm_h}, j_{h_1}>\ldots>j_{hm_h}\}$.

Then $\overline{j_{h1}} = \overline{j^1_{v_h}(w)}$ for some $1 < v_h < \ell$.

Proof: Let $E = \{\overline{j^1_v(w)} \mid 1 < v < \ell\} \subseteq \underline{n}$. Since $w \in \sigma^{-1}(\lambda)$, there exists $S = S_1 \cup \ldots \cup S_t$ satisfying $C_n(w,t)$ such that $|S| = \sum_{h=1}^{t} \lambda_h$ and $S_h = \{e(i_{hk}, j_{hk}) \mid 1 < k < m_h,\ i_{h1} < \ldots < i_{hm_h},\ j_{h1} > \ldots > j_{hm_h}\}$, $1 < h < t$. If $\{\overline{j_{h1}} \mid 1 < h < t\} \subseteq E$, then S is as required. Otherwise, there exists $\overline{j_{h1}} \notin E$, say $\overline{j_{h1}} = \overline{j^u_v(w)}$ for some $1 < v < \ell$, $u > 1$. Without loss of generality, we may assume that $j_{h1} = j^u_v(w)$. Let

$$\begin{cases} S^1_h = S_h \cup \{e((w),\ j^\alpha_v(w)) \mid 1 < \alpha < u\} \\ S^1_k = S_k - \{e(i,j) \mid \overline{j} \in \{\overline{j^\alpha_v(w)} \mid 1 < \alpha < u\}\},\ 1 < k < t,\quad k \neq h. \end{cases}$$

Then $S^1 = S^1_1 \cup \ldots \cup S^1_t$ satisfies $C_n(w,t)$ with $|S^1| > |S|$. But $|S^1| < \sum_{h=1}^{t} \lambda_h$ in general since $w \in \sigma^{-1}(\lambda)$. So we must have $|S^1| = |S|$.

For S, let $a(S,h) > 0$ be the largest number such that $e(i_{h\alpha}, j_{h\alpha}) = e((w),\ j^\alpha_v(w))$ for all $1 < \alpha < a(S,h)$ and some $1 < v < \ell$. Let $a(S) = \sum_{h=1}^{t} a(S,h)$. Similarly, we can define $a(S^1)$ for S^1. Then it is clear that $n > a(S^1) \gneq a(S)$. If S^1 is still not as required, then by the same procedure, we can get $S^2, S^3, \ldots$ in turn with $a(S^1) \lneq a(S^2) \lneq \ldots$. Since $a(S^\alpha) < n$ for all $\alpha > 1$. This implies that after a finite number of steps, we can get the required disjoint union of descending chains of w. $\square$

<u>Lemma 9.3.3</u> <u>Assume that $\tilde{w} \in \tilde{\sigma}^{-1}(\lambda)$ has a full DC form $(A_\ell,\ldots,A_1)$.</u> <u>Then for any $1 < t < r$, there exists $S = S_1 \cup \ldots \cup S_t$ satisfying</u> <u>$C_n(\tilde{w},t)$ and also satisfying the following conditions.</u>

(i) <u>Let $|S_h| = m_h$, $1 < h < t$. Then $\sum_{h=1}^{t} m_h = \sum_{h=1}^{t} \lambda_h$.</u>

(ii) <u>Let $S_h = \{e_{(hk)} \mid 1 < k < m_h,\ r(e_{(hk_1)}, e_{(hk_2)}) > 0,$</u>

<u>$c(e_{(hk_1)}, e_{(hk_2)}) < 0$, for any $1 < k_1 < k_2 < m_h\}$</u>

<u>Then $r(e_{(h1)}, e_{v_h 1}) = C(e_{(h1)}, e_{v_h 1}) \in n\mathbb{Z}$ for some $1 < v_h < \ell$,</u> <u>where e_{uv} is the v-th entry of A_u.</u>

<u>Proof</u>: This is only a version of Lemma 9.3.2 in $\tilde{A}_n$. □

<u>Proof of Lemma 9.3.1</u> Let $\tilde{y} \in \operatorname{Im} d(\lambda,m)$ with $\tilde{y} \in \tilde{\sigma}^{-1}_{n-rm}(\tau')$. Let $\tilde{w} \in \tilde{F}(\lambda,m)$ be such that $\tilde{y} = d(\lambda,m)(\tilde{w})$ and $\tilde{w}$ has a full MDC form $(A_r,\ldots,A_1)$ which is normal for the first m layers. Let $e_{uv}(\tilde{w})$ be the v-th entry of $A_u(\tilde{w})$, for u,v with $1 < u < r$, $1 < v < |A_u(\tilde{w})|$. First we claim that $\tau' > \tau$. For, suppose that $S = S_1 \cup \ldots \cup S_t$ satisfies $C_n(\tilde{w},t)$ with $|S| = \sum_{h=1}^{t} \lambda_h$, $1<t<r$. Let S' be obtained from S by deleting $\zeta(\tilde{w},m)$. Then by Lemma 8.5.14, $|S'| > |S| - tm = \sum_{h=1}^{t} \tau_h$. Since S' satisfies $C_{n-rm}(\tilde{y},t')$ with $t' < t$, we must have $\tau' > \tau$, where $t' = t$ if $\tau_t > 1$ or $t' = \max\{h \mid 1 < h < r,\ \tau_h > 1\}$ if $\tau_t = 0$.

Secondly, we claim that $\tau' < \tau$. For, by Lemma 9.3.3, for any $1 < t < r$, there exists $S' = S'_1 \cup \ldots \cup S'_t$ satisfying $C_{n-mr}(\tilde{y},t)$ and all conditions obtained from those in Lemma 9.3.3

by replacing λ, $\tilde{w}$ by τ', $\tilde{y}$. This implies that for any $1 \le t \le r$, there exists $S = S_1 \cup \ldots \cup S_t$ satisfying $C_n(\tilde{w},t)$ such that

(i) no entry of S lies in $\zeta(\tilde{w},m)$.

(ii) $|S| = \sum_{h=1}^{t} \tau'_h$.

(iii) Let $S_h = \{e_{(hk)} \mid 1 \le k \le |S_h|,\ r(e_{(hk_1)}, e_{(hk_2)}) > 0,$

$c(e_{(hk_1)}, e_{(hk_2)}) < 0$ for $1 \le k_1 < k_2 \le |S_h|\}$.

Then $r(e_{(h1)}, e_{u_h,m+1}(\tilde{w})) = c(e_{(h1)}, e_{u_h,m+1}(\tilde{w})) \in n\mathbb{Z}$ for some $u_h \in \{1,\ldots,r\}$.

Without loss of generality, we may assume $e_{(h1)} = e_{u_h,m+1}(\tilde{w})$ for $1 \le h \le t$. Let

$$S''_h = S_h \cup \{e_{u_h,\alpha}(\tilde{w}) \mid 1 \le \alpha \le m\},\ 1 \le h \le t.$$

Then $S'' = S''_1 \cup \ldots \cup S''_t$ satisfies $C_n(\tilde{w},t)$ with $|S''| = \sum_{h=1}^{t} \tau'_h + tm$. It follows by $\tilde{w} \in \tilde{\sigma}_n^{-1}(\lambda)$ that $|S''| \le \sum_{h=1}^{t} \lambda_h$. Hence $\sum_{h=1}^{t} \tau'_h \le \sum_{h=1}^{t} \tau_h$. So $\tau' \le \tau$ and then $\tau = \tau'$, i.e. $\tilde{y} \in \tilde{\sigma}_{n-rm}^{-1}(\tau)$. Im $d(\lambda,m) \subseteq \tilde{\sigma}_{n-rm}^{-1}(\tau)$.

□

§9.4 THE SUBSET $\tilde{H}_{\lambda,k}$ OF $\tilde{\sigma}^{-1}(\lambda)$

In this section, we introduce a certain subset $\tilde{H}_{\lambda,k}$ of $\tilde{A}_n$ and show that certain combinations of ρ operations can be applied to any given element $\tilde{w} \in \tilde{H}_{\lambda,k}$. This will subsequently enable us to construct the sequence $\xi(\tilde{w},k)$.

In this section and §9.5, §9.6, we shall assume that $1 \le k < r$ and $\lambda_k \geqslant \lambda_{k+1}$.

Let $k = i_o < i_1 < \ldots < i_t = r$ and $i_{t+1} = r+1$ be such that for every $0 \leq h \leq t$, $\lambda_{i_h} \geq \lambda_{i_h+1} = \ldots = \lambda_{i_{h+1}}$. Let $\alpha_{\lambda,h} = \lambda_{i_h} - \lambda_{i_{h+1}}$, $\tau_a^h = \lambda_a - \lambda_{i_{h+1}}$, $\beta_{\lambda,h} = n - \lambda_{i_{h+1}} \sum_{j=1} \mu_j$ for h, a with $0 \leq h \leq t$ and $1 \leq a \leq i_h$. We have $\tau^h = \{\tau_1^h \geq \ldots \geq \tau_{i_h}^h\} \in \Lambda_{\beta_{\lambda,h}}$.

Let $\tilde{N}_\tau = \eta(N_\tau)$ for any $\tau \in \Lambda_m$ with $m \geq 1$. Given $\tilde{y} \in \tilde{N}_{\tau^h}$ with $1 \leq h < t$, we construct an element $\tilde{w}$ as follows.

Assume that $\tilde{y}$ has the standard MDC form $(B_{i_h},\ldots,B_1)$. Let $f_{pq}(\tilde{y})$ be the row of $\tilde{y}$ containing the entry $e_{pq}(\tilde{y})$. For each u, $0 \leq u < h$, let $B_{i_u}^o(\tilde{y})$ (resp. $B_{i_u}^1(\tilde{y})$) be the block consisting of the first $\tau_{i_{u+1}}^h$ (resp. the last $\tau_{i_u}^h - \tau_{i_{u+1}}^h - 1$) rows of $B_{i_u}(\tilde{y})$. We permute the blocks from $[B_{i_{u+1}-1}, B_{i_{u+1}-2},\ldots,B_{i_u+1},B_{i_u}^o, f_{i_u,\tau_{i_{u+1}}^h+1}]$ to $[f_{i_u,\tau_{i_{u+1}}^h+1}, B_{i_{u+1}-1},\ldots,B_{i_u+1},B_{i_u}^o]$ in the matrix $\tilde{y}$ for each u, $0 \leq u < h$, and get $\tilde{w}$.

Let $\tilde{H}_{\tau^h,k}$ be the set of all elements $\tilde{w}$ of $\tilde{A}_{\beta_{\lambda,h}}$ obtained by the above procedure. We see that for any $\tilde{w} \in \tilde{H}_{\tau^h,k}$ the corresponding $\tilde{y} \in \tilde{N}_{\tau^h}$ must be uniquely determined and also the standard MDC form $(B_{i_h},\ldots,B_1)$ of $\tilde{y}$ is uniquely determined.

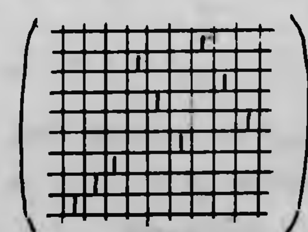
Let $\tilde{w} \in \tilde{H}_{\tau^h,k}$ have full MDC form $(A_{i_h},\ldots,A_1)$. We say that this MDC form of $\tilde{w}$ is semistandard if it comes from a standard MDC form $(B_{i_h},\ldots,B_1)$ of $\tilde{y} \in \tilde{N}_{\tau^h}$ by the above procedure, where

$$A_v(\tilde{w}) \text{ comes from} \begin{cases} B_v(\tilde{y}) \text{ if } 1 < v < i_h,\ v \notin \{i_o,\dots,i_h\}. \\ [B^o_{i_u}(\tilde{y}),\ B^1_{i_u}(\tilde{y}),\ f_{i_{u-1},\tau^h_{i_u}+1}(\tilde{y})] \text{ if } v = i_u,\ 0 < u < h. \\ [B^o_{i_o}(\tilde{y}),\ B^1_{i_o}(\tilde{y})] \text{ if } v = i_o. \\ [B_{i_h}(\tilde{y}),\ f_{i_{h-1},\tau^h_{i_h}+1}(\tilde{y})] \text{ if } v = i_h. \end{cases}$$

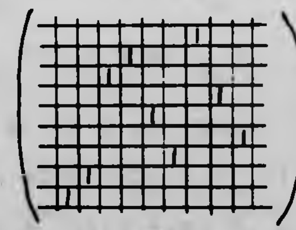
Clearly, the semistandard MDC form of $\tilde{w}$ is uniquely determined.

Let $\tilde{H}_{\tau^o,k} = \tilde{N}_{\tau^o}$. Clearly, $\tilde{H}_{\tau^h,k} \subseteq \tilde{F}(\tau^h, \tau^h_{i_h})$ for all $0 < h < t$ and $\tilde{H}_{\lambda,k} = \tilde{H}_{\tau^t,k}$.

Let us give an example for an element $\tilde{w} \in \tilde{A}_n$ which lies in $\tilde{H}_{\lambda,k}$ and is obtained from $\tilde{y} \in \tilde{N}_\lambda$ in the above way, where $n = 9$, $k = 1$, $\lambda = \{5 > 2 > 2\}$.



$\tilde{y}$



$\tilde{w}$

<u>Lemma 9.4.2</u> $\underline{\tilde{H}_{\lambda,k} \subseteq \tilde{\sigma}^{-1}(\lambda)}$.

<u>Proof</u>: Take any $\tilde{w} \in \tilde{H}_{\lambda,k}$. In the matrix $\tilde{w}$, let

$$\tilde{e}_{uv} = \begin{cases} e_{uv} & \text{for (i) } 1 < u < r,\ u \notin \{i_1,\dots,i_{t-1}\} \text{ (ii) } u = i_h \text{ and } 1 < v < \lambda_{i_{h+1}} \text{ with } 0 < h < t-1. \\ e_{u,v-1} & \text{for (i) } u = i_h,\ 1 < h < t,\ \lambda_{i_{h+1}}+1 < v < \lambda_{i_h}+1 \text{ (ii) } u = i_o,\ \lambda_{i_1} + 1 < v < \lambda_{i_o}. \end{cases}$$

Then $0 < c(\tilde{e}_{uv},\tilde{e}_{u'v}) < n$ for $1 \le u' < u \le r$ and $1 \le v \le \lambda_1$ when $\tilde{e}_{uv}$, $\tilde{e}_{u'v}$ both exist.

Suppose that $\tilde{w} \in \tilde{\sigma}^{-1}(\lambda')$, $\lambda' = \{\lambda'_1 \ge \dots \ge \lambda'_{r'}\}$. Let $S = S_1 \cup \dots \cup S_h$ satisfy $C_n(\tilde{w},h)$. Let $\bar{e}_v$ be the set of entry classes containing all $\tilde{e}_{uv}$, $1 \le u \le r$, which exist, and let $\mu_v = |\bar{e}_v|$. Then by Lemma 8.5.14, $|S \cap \bar{e}_v| \le \min\{\mu_v, h\}$. This implies

$$|S| = \sum_{v=1}^{\lambda_1} |S \cap \bar{e}_v| \le \sum_{i=1}^{h} \lambda_i$$

So $\lambda' \le \lambda$.

On the other hand, let $S'_v = \{\text{all entries in } A_v(\tilde{w})\}$, $1 \le v < k$, $S'_k = \left(\bigcup_{h=1}^{t-1} \{\tilde{e}_{i_h,v}(\tilde{w}) \mid \lambda_{i_{h+1}} + 1 \le v \le \lambda_{i_h} + 1\}\right) \cup \{\tilde{e}_{i_t,v} \mid 1 \le v \le \lambda_{i_t} + 1\} \cup \{\tilde{e}_{i_o,v} \mid \lambda_{i_1} + 1 \le v \le \lambda_{i_o}\}$, $S'_h = \{\text{all entries in } A_{h-1}(\tilde{w})\} - S'_k$, $k < h \le r$. Then $S' = S'_1 \cup \dots \cup S'_r$ satisfies $C_n(\tilde{w},r)$ with $|S'_h| = \lambda_h$, $1 \le h \le r$. So $\lambda' \ge \lambda$. Therefore $\lambda = \lambda'$ and $\tilde{w} \in \tilde{\sigma}^{-1}(\lambda)$, i.e. $\tilde{H}_{\lambda,k} \subseteq \tilde{\sigma}^{-1}(\lambda)$. □

For any $\tilde{w} \in \tilde{H}_{\lambda,k}$ with semi-standard MDC form $(A_r,\dots,A_1)$, let $\tilde{v}_1 = d(\tau^t,\tau^t_{i_t})(\tilde{w})$. Then $\tilde{v}_1$ has the full MDC form $(A_{i_{t-1}},\dots,A_1)$ satisfying all conditions for $\tilde{H}_{\tau^{t-1},k}$, where $A_{i_{t-1}}(\tilde{v}_1)$ is the MDC block of $\tilde{v}_1$ coming from $[A_{i_t}(\tilde{w}),\dots,A_{i_{t-1}}(\tilde{w})]$ by deletion and $A_u(\tilde{v}_1)$ comes from $A_u(\tilde{w})$ for $1 \le u < i_{t-1}$. So $\tilde{v}_1 \in \tilde{H}_{\tau^{t-1},k}$, $d(\tau^t,\tau^t_{i_t})(\tilde{H}_{\lambda,k}) \subseteq \tilde{H}_{\tau^{t-1},k}$. In general, for any $1 \le h \le t$, $\tilde{v}_{t+1-h} = d(\tau^h,\tau^h_{i_h}) \cdot d(\tau^{h+1},\tau^{h+1}_{i_{h+1}}) \dots d(\tau^t,\tau^t_{i_t})(\tilde{w})$ exists and has the full MDC form $(A_{i_{h-1}},\dots,A_1)$ satisfying all conditions for $\tilde{H}_{\tau^{h-1},k}$. Then $\tilde{v}_{t+1-h} \in \tilde{H}_{\tau^{h-1},k}$, where $A_{i_{h-1}}(\tilde{v}_{t+1-h})$ comes from $[A_{i_h}(\tilde{v}_{t-h}),\dots,A_{i_{h-1}}(\tilde{v}_{t-h})]$ and $A_u(\tilde{v}_{t+1-h})$ comes from $A_u(\tilde{v}_{t-h})$, $1 < u < i_{h-1}$. Finally, there exists $\tilde{v}_t = d(\tau',\tau'_{i_1})(\tilde{v}_{t-1})$ having the full MDC form $([A_{k+1},A_k], A_{k-1},\dots,A_1)$ satisfying all conditions for $\tilde{H}_{\tau^0}$, where $A_{k+1}(\tilde{v}_t)$ comes from $[A_{i_1}(\tilde{v}_{t-1}),\dots,A_{k+1}(\tilde{v}_{t-1})]$ and $A_u(\tilde{v}_t)$ comes from $A_u(\tilde{v}_{t-1})$, $1 < u < k$. Thus we have

$$\tilde{H}_{\tau^t,k} \xrightarrow{d(\tau^t,\tau^t_{i_t})} \tilde{H}_{\tau^{t-1},k} \xrightarrow{d(\tau^{t-1},\tau^{t-1}_{i_{t-1}})} \cdots \xrightarrow{d(\tau^2,\tau^2_{i_2})} \tilde{H}_{\tau^1,k} \xrightarrow{d(\tau^1,\tau^1_{i_1})} \tilde{H}_{\tau^0,k}$$

Under the map $\tilde{w} \to d(\tau^h,\tau^h_{i_h})\cdot d(\tau^{h+1},\tau^{h+1}_{i_{h+1}})\dots d(\tau^t,\tau^t_{i_t})(\tilde{w})$ the blocks of $\tilde{w}$ are deleted into the following ones.

$$[A_r(\tilde{w}),\dots,A_{k+1}(\tilde{w})] \to \begin{cases} [A_{i_{h-1}}(\tilde{v}_{t+1-h}),\dots,A_{k+1}(\tilde{v}_{t+1-h})] & h > 1 \\ A_{k+1}(\tilde{v}_t) & h = 1 \end{cases}$$

$$[A_r(\tilde{w}),\dots,A_{i_1}(\tilde{w})] \to \begin{cases} [A_{i_{h-1}}(\tilde{v}_{t+1-h}),\dots,A_{i_1}(\tilde{v}_{t+1-h})] & h > 1 \\ A_{k+1}(\tilde{v}_t) & h = 1 \end{cases}$$

$$[A_r(\tilde{w}),\dots A_{i_1+1}(\tilde{w})] \to \begin{cases} [A_{i_{h-1}}(\tilde{v}_{t+1-h}),\dots,A_{i_1+1}(\tilde{v}_{t+1-h})] & h > 2 \\ A_{i_1,1}(\tilde{v}_{t-1}) & h = 2 \\ \emptyset & h = 1 \end{cases}$$

$$A^{o}_{i_1}(\tilde{w}) \longrightarrow \begin{cases} A^{o}_{i_1}(\tilde{v}_{t+1-h}) & h > 2 \\ A^{*}_{i_1}(\tilde{v}_{t-1}) & h = 2 \\ \emptyset & h = 1 \end{cases}$$

$$A^{1}_{i_1}(\tilde{w}) \longrightarrow \begin{cases} A^{1}_{i_1}(\tilde{v}_{t+1-h}) & h > 1 \\ A_{k+1}(\tilde{v}_t) & h = 1 \end{cases}$$

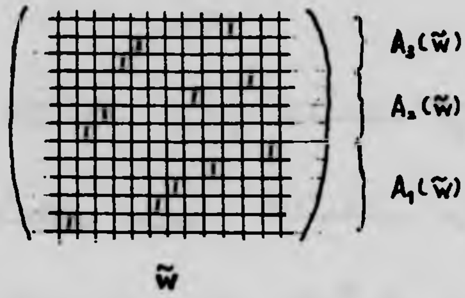
$$A_{i_1}(\tilde{w}) \longrightarrow \begin{cases} A_{i_1}(\tilde{v}_{t+1-h}) & h > 2 \\ A'_{i_1,|A_{i_1}|-1}(\tilde{v}_{t-1}) & h = 2 \\ A_{k+1}(\tilde{v}_t) & h = 1 \end{cases}$$

$$A_u(\tilde{w}) \longrightarrow A_u(\tilde{v}_{t+1-h})$$
$(1 < u < k)$

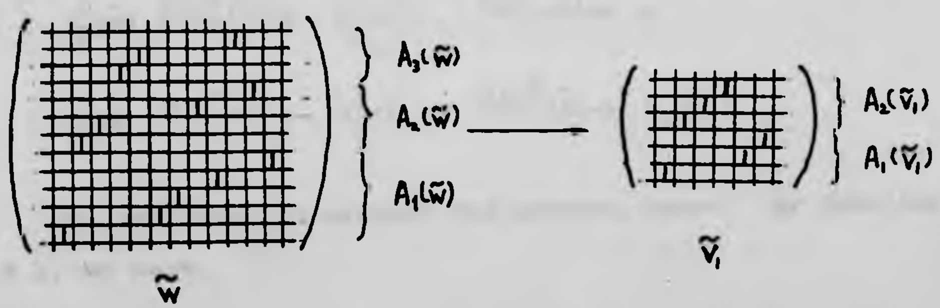
where $A_{i_1,u}$ (resp. $A'_{i_1,u}$) is the block consisting of the first u (resp. the last u) rows of A_{i_1} for $0 < u < |A_{i_1}|$. $A^{o}_{i_1} = A_{i_1,|A_{i_1}|-1}$. $A^{*}_{i_1}$ is the block obtained from A_{i_1} by omitting the first row and the last row. $\emptyset$ is the block of size 0. $A^{1}_{i_1} = A'_{i_1,1}$.

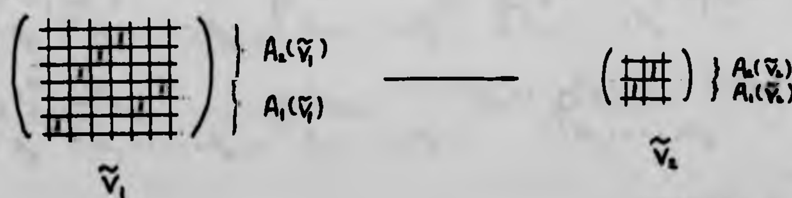
Now let us observe an example to illustrate what the above statement means.

Suppose $\tilde{w} \in \tilde{H}_{\lambda,1}$ with $\lambda = (6 > 4 > 2) \in \Lambda_{12}$ which has the following semi-standard MDC form (A_3, A_2, A_1).



So by using the above notation, we have $k = 1$, $t = 2$, $i_1 = 2$. Let $\tau^1 = \{4 > 2\} \in \Lambda_6$ and $\tau^0 = \{2\} \in \Lambda_2$. Then $\tilde{v}_1 = d(\tau^1,2)(\tilde{w}) \in \tilde{H}_{\tau^1,1}$ and $\tilde{v}_2 = d(\tau^0,2)(\tilde{v}_1) \in \tilde{H}_{\tau^0,1} = \tilde{N}_{\tau^0}$ are as follows.





where each matrix on the right is obtained from the corresponding one on the left by leaving out all rows and columns fully shadowed.

We see that the blocks of $\tilde{w}$ here are deleted into the following ones.

$$[A_3(\tilde{w}),\ A_2(\tilde{w})] \xrightarrow{d(\tau^1,2)} A_2(\tilde{v}_1) \xrightarrow{d(\tau^0,2)} A_2(\tilde{v}_2)$$

$$A_1(\tilde{w}) \xrightarrow{d(\tau^1,2)} A_1(\tilde{v}_1) \xrightarrow{d(\tau^0,2)} A_1(\tilde{v}_2)$$

$$A_3(\tilde{w}) \xrightarrow{d(\tau^1,2)} A_{2,1}(\tilde{v}_1) \xrightarrow{d(\tau^0,2)} \emptyset$$

$$A_2(\tilde{w}) \xrightarrow{d(\tau^1,2)} A^{!}_{2,2}(\tilde{v}_1) \xrightarrow{d(\tau^0,2)} A_2(\tilde{v}_2)$$

$$A_2^0(\tilde{w}) \xrightarrow{d(\tau^1,2)} A_2^*(\tilde{v}_1) \xrightarrow{d(\tau^0,2)} \emptyset$$

$$A_2^1(\tilde{w}) \xrightarrow{d(\tau^1,2)} A_2^1(\tilde{v}_1) \xrightarrow{d(\tau^0,2)} A_2(\tilde{v}_2)$$

Now we return to discuss the general cases. By Corollary 8.8.2, we have

$$\rho^{A^0_{i_1}}_{A_k} \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}} \rho^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_k} (\rho^{A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{w}) \text{ exists}$$

$$\Leftrightarrow \rho^{A^0_{i_1}}_{A_k} \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}} \rho^{A_{i_{h-1}},\ldots,A_{i_1+1}}_{A_1,\ldots,A_k} (\rho^{A_{i_{h-1}},\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_{t+1-h})$$

exists, $2 < h < t$.

$$\Leftrightarrow \rho^{A^*_{i_1}}_{A_k} \rho^{A^{!}_{i_1},|A_{i_1}|-1}_{A_1,\ldots,A_{k-1}} \rho^{A_{i_1},1}_{A_1,\ldots,A_k} (\rho^{A_{i_1},\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_{t-1})$$

$$\Leftrightarrow \rho^{A^0_{i_1}}_{A_k} \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}} (\rho^{A_{i_1},\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_{t-1}) \text{ exists} \qquad (1)$$

Also,

$$\rho^{A_r,\ldots,A_{i_1}}_{A_1,\ldots,A_k}(\rho^{A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{w}) \text{ exists}$$

$$\longleftrightarrow \rho^{A_{i_{h-1}},\ldots,A_{i_1}}_{A_1,\ldots,A_k}(\rho^{A_{i_{h-1}},\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_{t+1-h}) \text{ exists } 1 < h < t$$

$$\longleftrightarrow (\rho^{A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}}(\tilde{v}_t) \text{ exists.} \qquad (2)$$

Also,

$$\rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}(\rho^{A_{i_1},\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_{t-1}) \text{ exists}$$

$$\longleftrightarrow \rho^{A_{k+1}}_{A_1,\ldots,A_{k-1}}(\rho^{A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_t) \text{ exists} \qquad (3)$$

Since $\tilde{v}_{t,1} = \rho^{A_{k+1}}_{A_1,\ldots,A_{k-1}}(\rho^{A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_t)$ is the last term of the sequence $\xi(\tilde{v}_t,k)$ beginning with $\tilde{v}_t \in H_{\tau^0}$, we see that $\tilde{v}_{t,1}$ exists but $\rho^{A_{k+1}}_{A_k}(\tilde{v}_{t,1})$ doesn't. It follows from (2), (3) that $\tilde{v}_{t-1,1} = \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}(\rho^{A_{i_1},\ldots,A_{k+1}}_{A_1,\ldots,A_k})^{\alpha_{\lambda,0}^{-1}}(\tilde{v}_{t-1})$ exists but $\rho^{A_{i_1}}_{A_k}(\tilde{v}_{t-1,1})$ doesn't, where we stipulate $\tilde{v}_o = \tilde{w}$.

We claim that $\rho^{A^o_{i_1}}_{A_k}(\tilde{v}_{t-1,1})$ exists. Otherwise, there exists u with $0 < u < |A_{i_1}| - 1$ such that $\tilde{v}_{t-1,2} = \rho_{A_k}^{A_{i_1},u}(\tilde{v}_{t-1,1})$ exists but $\rho_{A_k}^{A_{i_1},u+1}(\tilde{v}_{t-1,1})$ doesn't. Hence $\tilde{v}_{t-1,2}$ has the DC form $(A_{i_1-1},\dots,A_{k+1}, A_{i_1,u}, [A_k, A'_{i_1,|A_{i_1}|-u}], A_{k-1},\dots,A_1)$ with $|[A_k(\tilde{v}_{t-1,2}), A'_{i_1,|A_{i_1}|-u}(\tilde{v}_{t-1,2})]| > \tau_k^1 + 1$ and then

$$\sum_{u=1}^{k-1} |A_u(\tilde{v}_{t-1,2})| + |[A_k(\tilde{v}_{t-1,2}), A'_{i_1,|A_{i_1}|-u}(\tilde{v}_{t-1,2})]| > \sum_{u=1}^{k} \tau_u^1 + 1.$$

But by Lemmas 9.3.1 and 3.7, $\tilde{v}_{t-1,2} \in \tilde{\sigma}^{-2}_{\beta_{\lambda,1}}(\tau^1)$. This gives a contradiction.

Therefore, by (1), (2), we have

$$\tilde{w}' = \rho_{A_k}^{A^o_{i_1}} \rho_{A_1,\dots,A_{k-1}}^{A_{i_1}} \rho_{A_1,\dots,A_k}^{A_r,\dots,A_{i_1+1}} \left(\rho_{A_1,\dots,A_k}^{A_r,\dots,A_{k+1}}\right)^{\alpha_{\lambda,0}-1} (\tilde{w}) \text{ exists}$$

but $\rho_{A_k}^{A^1_{i_1}}(\tilde{w}')$ doesn't. This means that $\tilde{w}'$ has the full MDC form

$(A_{i_1-1},\dots,A_{k+1}, A_r,\dots,A_{i_1+1}, A^o_{i_1}, [A_k, A^1_{i_1}], A_{k-1},\dots,A_1)$

where when $t > 1$,

$$\begin{cases} |A_u| = \lambda_u, & 1 < u < r, \quad u \neq k, i_1 \\ |A^o_{i_1}| = \lambda_{i_1} - 1 & \\ |[A_k, A^1_{i_1}]| = \lambda_k & \\ |A_r| = \lambda_r + 1 & \end{cases}$$

When $t = 1$, $\begin{cases} |A_u| = \lambda_u, & 1 < u < i_1, \ u \neq k \\ |A^o_{i_1}| = \lambda_{i_1} & \\ |[A_k, A^1_{i_1}]| = \lambda_k. & \end{cases}$

§9.5 THE SEQUENCE $\xi(\tilde{w},k)$ BEGINNING WITH $\tilde{w} \in \tilde{N}_\lambda$.

For any $\tilde{w} \in \tilde{N}_\lambda$, assume that $\tilde{w}$ has the standard MDC form $(A_r,\ldots,A_1)$. We split $A_k(\tilde{w})$ into two blocks $A_k^o(\tilde{w})$ and $A_k^1(\tilde{w})$, where $A_k^o(\tilde{w})$ (resp. $A_k^1(\tilde{w})$) consists of the first row (resp. the last λ_k-1 rows) of $A_k(\tilde{w})$. By Lemmas 8.5.10 and 8.5.12, there exists $\tilde{w}' = \rho^{A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1}}(\tilde{w})$ having full MDC form

$(A_k,A_r,A_{r-1},\ldots,A_{k+1},A_{k-1},\ldots,A_1)$ which is normal. We claim that there also exists $\tilde{w}'' = \rho^{A_r,\ldots,A_{k+1}}_{A_k^1}(\tilde{w}')$ since $\lambda_k \geqslant \lambda_h$, $k+1 < h < r$ and $\tilde{w}'$ has local MDC form $(A_k^1,A_r,\ldots,A_{k+1})$ which is quasi-normal for the first λ_k-1 layers. Then $\tilde{w}''$ has full MDC form $([A_k^o,A_r],\ A_{r-1},\ldots,A_{k+1},A_k^1,A_{k-1},\ldots,A_1)$ satisfying all conditions for $\tilde{H}_{\lambda,k}$. i.e. $\tilde{w}'' \in \tilde{H}_{\lambda,k}$.

Combining the above result with §9.4 and noting that $[A_k^o,A_r]$, A_k^1 play the same roles as A_r, A_k in §9.4, we see that there exists

$$\tilde{y} = \rho^{A_{i_1}^o}_{A_k^1}\rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}\rho^{A_k^o,A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}\left(\rho^{A_k^o,A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1},A_k^1}\right)^{\alpha_{\lambda,0}-1}$$

$$\rho^{A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1},A_k^1}(\tilde{w}) =$$

$$= \rho^{A_{i_1}^o}_{A_k^1}\rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}\rho^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}\left(\rho^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}\right)^{\alpha_{\lambda,0}}(\tilde{w}) \qquad (1)$$

but there doesn't exist $\rho^{A_{i_1}^1}_{A_k^1}(\tilde{y})$. So there exists a sequence $\tilde{w}_o = \tilde{w},\ \tilde{w}_1,\ldots,\tilde{w}_{\alpha_{\lambda,0}+3} = \tilde{y}$ such that

$$\begin{cases} \tilde{w}_u = \rho^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}(\tilde{w}_{u-1}) & \text{for } 1 < u < \alpha_{\lambda,0} \\ \tilde{w}_{\alpha_{\lambda,0}+1} = \rho^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}(\tilde{w}_{\alpha_{\lambda,0}}) & \\ \tilde{w}_{\alpha_{\lambda,0}+2} = \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}(\tilde{w}_{\alpha_{\lambda,0}+1}) & \\ \tilde{w}_{\alpha_{\lambda,0}+3} = \rho^{A^o_{i_1}}_{A_k^1}(\tilde{w}_{\alpha_{\lambda,0}+2}) & \end{cases}$$

By linking all the sequences $\xi(\tilde{w}_u, \rho^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1})$, $0 < u < \alpha_{\lambda,0}$, $\xi(\tilde{w}_{\alpha_{\lambda,0}}, \rho^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1})$, $\xi(\tilde{w}_{\alpha_{\lambda,0}+1}, \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}})$ and $\xi(\tilde{w}_{\alpha_{\lambda,0}+2}, \rho^{A^o_{i_1}}_{A_k^1})$ together (for the definition of these sequences, see §5.4, §8.4), we get the sequence $\xi(\tilde{w},k)$ beginning with $\tilde{w} \in \tilde{N}_\lambda$.

<u>Lemma 9.5.2 The length of $\xi(\tilde{w},k)$ is independent of the choice of $\tilde{w}$ in $\tilde{N}_\lambda$.</u>

<u>Proof</u> This follows from formula (1). □

Let $\tilde{x}_o = \tilde{w}, \tilde{x}_1,\ldots,\tilde{x}_{\alpha_{\lambda,0}+1}, \tilde{z}$ be such that for $1 < j < \alpha_{\lambda,0} + 1$,

$$\begin{cases} \tilde{x}_j = \rho^{A^0_{i_1}}_{A^1_k}\, \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}\, \rho^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A^1_k} \left(\rho^{A_r,\ldots,A_{k+1},A^0_k}_{A_1,\ldots,A_{k-1},A^1_k}\right)^{j-1}(\tilde{w}) \\ \\ \tilde{z} = \rho^{A_r,\ldots,A_{k+1},A^0_k}_{A_1,\ldots,A_{k-1},A^1_k}(\tilde{w}) \end{cases}$$

Then all $\tilde{x}_j$, $0 < j < \alpha_{\lambda,0} + 1$, $\tilde{z}$ lie in $\xi(\tilde{w},k)$. Let us consider full MDC forms for these elements.

(i) Suppose $t > 1$.

Since $\rho^{A_r}_{A_1,\ldots,A_{k-1},A^1_k}(\tilde{w})$ has full MDC form $(A_{r-1},\ldots,A_{k+1},\ [A^0_k,A_r],\ A^1_k,A_{k-1},\ldots,A_1)$ and

$$\tilde{x}_j = \rho^{A^0_{i_1}}_{A^1_k}\, \rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}\, \rho^{A_{r-1},\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A^1_k} \left(\rho^{A_{r-1},\ldots,A_{k+1},[A^0_k,A_r]}_{A_1,\ldots,A_{k-1},A^1_k}\right)^{j-1}$$

$\rho^{A_r}_{A_1,\ldots,A^1_k}(\tilde{w})$ for $1 < j < \alpha_{\lambda,0} + 1$, we see that $[A^0_k(\tilde{x}_j), A_r(\tilde{x}_j)]$ is an MDC block of $\tilde{x}_j$ for $1 < j < \alpha_{\lambda,0} + 1$. Since $\rho^{A^1_{i_1}}_{A^1_k}(\tilde{x}_j)$ exists for $1 < j < \alpha_{\lambda,0}$ but $\rho^{A^1_{i_1}}_{A^1_k}(\tilde{x}_{\alpha_{\lambda,0}+1})$ doesn't, we see that $[A^1_k(\tilde{x}_j),\ A^1_{i_1}(\tilde{x}_j)]$ is not a DC block of $\tilde{x}_j$ for $1 < j < \alpha_{\lambda,0}$ but $[A^1_k(\tilde{x}_{\alpha_{\lambda,0}+1}),\ A^1_{i_1}(\tilde{x}_{\alpha_{\lambda,0}+1})]$ is a DC block of $\tilde{x}_{\alpha_{\lambda,0}+1}$. So by Lemma 5.2.3, $\tilde{x}_j$ has full MDC form

$(A_{i_1-1},\ldots,A_{k+1},[A^0_k,A_r],A_{r-1},\ldots,A_{i_1+1},A^0_{i_1},A^1_k,A^1_{i_1},A_{k-1},\ldots,A_1)$

for $1 < j < \alpha_{\lambda,0}$ and $\tilde{x}_{\alpha_{\lambda,0}+1}$ has full MDC form

$(A_{i_1-1},\ldots,A_{k+1},[A^0_k,A_r],A_{r-1},\ldots,A_{i_1+1},A^0_{i_1},[A^1_k,A^1_{i_1}],A_{k-1},\ldots,A_1)$

(ii) Suppose $t = 1$.

The results on $[A^1_k, A^1_{i_1}]$ are the same as that in the case $t > 1$.

Since $\rho^{A^o_{i_1}}_{A_1,\dots,A_{k-1},A^1_k}(\tilde{w})$ has full MDC form

$(A^1_{i_1}, A_{i_1-1},\dots,A_{k+1}, [A^o_k, A^o_{i_1}], A^1_k, A_{k-1},\dots,A_1)$ and

$$\tilde{x}_j = \rho^{A^1_{i_1}}_{A_1,\dots,A_{k-1}}\left(\rho^{A^1_{i_1},A_{i_1-1},\dots,A_{k+1},[A^o_k,A^o_{i_1}]}_{A_1,\dots,A_{k-1},A^1_k}\right)^{j-1}\rho^{A^o_{i_1}}_{A_1,\dots,A_{k-1},A^1_k}(\tilde{w})$$

for $1 < j < \alpha_{\lambda,0}+1$, we see by Lemma 5.2.3 that $\tilde{x}_j$ has full MDC form $(A_{i_1-1},\dots,A_{k+1}, [A^o_k, A^o_{i_1}], A^1_k, A^1_{i_1}, A_{k-1},\dots,A_1)$ for $1 < j < \alpha_{\lambda,0}$, and $\tilde{x}_{\alpha_{\lambda,0}+1}$ has full MDC form

$(A_{i_1-1},\dots,A_{k+1}, [A^o_k, A^o_{i_1}], [A^1_k, A^1_{i_1}], A_{k-1},\dots,A_1)$.

On the other hand, since $\tilde{x} = \rho^{A_r,\dots,A_{k+1}}_{A_1,\dots,A_{k-1},A^1_k}(\tilde{w}) \in \tilde{H}_{\lambda,k}$ has full MDC form $([A^o_k, A_r], A_{r-1},\dots,A_{k+1}, A^1_k, A_{k-1},\dots,A_1)$ which is normal for the first λ_r layers and so has full local MDC form $(A_r,\dots,A_{k+1}, A^1_k, A_{k-1},\dots,A_1, A^o_k)$ which is quasi-normal for the first layer, we see by Lemma 8.5.12 that

$\tilde{z} = \rho^{A^o_k}_{A_1,\dots,A_{k-1},A^1_k}(\tilde{x})$ has full MDC form

$(A_r,\dots,A_{k+1}, A^o_k, A^1_k, A_{k-1},\dots,A_1)$ which is quasi-normal for the first layer. So we have

<u>Lemma 9.5.3</u> $\underline{\pi(\mathcal{L}(\tilde{x}_{\alpha_{\lambda,0}+1})) \geqslant \pi(\mathcal{L}(\tilde{x}_j)) = \pi(\mathcal{L}(\tilde{x}_{j'}))}$ for any $\underline{1 < j,\ j' < \alpha_{\lambda,0}.\ \ \pi(\mathcal{L}(\tilde{z})) = \{\lambda_1 > \ldots > \lambda_{k-1} > \lambda_k - 1 > \lambda_{k+1} > \ldots}$ $\underline{> \lambda_r > 1\}.}$ □

§9.6 THE SEQUENCE $\xi(w,k)$ BEGINNING WITH $w \in \mathbb{N}_\lambda$.

Assume that $w \in \mathbb{N}_\lambda$ has the standard MDC form $(A_r,\ldots,A_1)$ at $i \in \mathbb{Z}$. A_k^o, A_k^1, $A_{i_1}^o$, $A_{i_1}^1$ are defined in the same way as in §9.4. Then by Lemma 8.9.1, and §9.5, there exists

$$y = \wp^{A_{i_1}^o}_{A_k^1}{}^1\, \wp^{A_{i_1}}_{A_1,\ldots,A_{k-1}}{}^1\, \wp^{A_k^o,A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}{}^1 \left(\wp^{A_k^o,A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1},A_k^1}\right)^{\alpha_{\lambda,0}-1} \wp^{A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1},A_k^1}{}^1(w)$$

$$= \wp^{A_{i_1}^o}_{A_k^1}{}^1\, \wp^{A_{i_1}}_{A_1,\ldots,A_{k-1}}{}^1\, \wp^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1} \left(\wp^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}\right)^{\alpha_{\lambda,0}}(w)$$

but $\wp^{A_{i_1}^1}_{A_k^1}{}^1(y)$ doesn't exist. Then we can define the sequence $\xi(w,k)$ beginning with w in the same way as that in §9.5 and conclude that the length of $\xi(w,k)$ is independent of the choice of w in $\mathbb{N}_\lambda$. We can also define $x_o = w$, $x_1,\ldots,x_{\alpha_{\lambda,0}+1}$, z be such that for $1 < j < \alpha_{\lambda,0}+1$,

$$\begin{cases} x_j = \wp^{A_{i_1}^o}_{A_k^1}{}^1\, \wp^{A_{i_1}}_{A_1,\ldots,A_{k-1}}{}^1\, \wp^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}{}^1 \left(\wp^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}\right)^{j-1}(w) \\ z = \wp^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}{}^1(w) \end{cases}$$

and assert that all x_j, $0 < j < \alpha_{\lambda,0}+1$, z lie in $\xi(w,k)$.

We also have

Lemma 9.6.1 $\pi(\mathcal{L}(x_{\alpha_{\lambda,0}+1})) \gneqq \pi(\mathcal{L}(x_j)) = \pi(\mathcal{L}(x_{j'}))$ for any $1 \leq j,\ j' \leq \alpha_{\lambda,0}$, $\pi(\mathcal{L}(z)) = \{\lambda_1 \geq \ldots \geq \lambda_{k-1} \geq \lambda_k - 1 \geq \lambda_{k+1} \geq \ldots \geq \lambda_r \geq 1\}$

□

§9.7 INCREASING CHAINS

For $w \in N_\lambda$, assume that w has the standard MDC form $(A_r, \ldots, A_1)$ at $i \in \mathbb{Z}$. Let $E_u = \{\overline{j_h^u(w)} \mid 1 \leq h \leq \mu_u\} \subseteq \underline{n}$, $1 \leq u \leq \lambda_1$. Let $\zeta(E_u)$ be the set of all columns g of an affine matrix such that g are labelled by integers which, when taken mod n, lie in E_u. Let $\bar{e}((x), E_u)$ be the set of entries of x lying in $\zeta(E_u)$. Then for any e_1, $e_2 \in \bar{e}((w), E_u)$, we have $r(e_1,e_2)\cdot c(e_1,e_2) \geq 0$.

Definition 9.7.1 For any $w \in A_n$, we call a set $\bar{e}$ of entries of w an increasing chain if $\bar{e}$ is a union of some entry classes and for e_1, $e_2 \in \bar{e}$, $r(e_1,e_2)\cdot c(e_1,e_2) \geq 0$ holds.

It is easily seen that $\bar{e}$ is an increasing chain if and only if for any $e_1, e_2 \in \bar{e}$ with $0 < r(e_1,e_2) < n$, we have $0 < c(e_1,e_2) < n$.

By this definition, the sets $\bar{e}((w),E_u)$, $1 \leq u \leq \lambda_1$ are all increasing chains of w. Now we shall consider $\bar{e}((x),E_u)$, $1 \leq u \leq \lambda_1$, where x is some special element lying in the sequence $\xi(w,k)$ for $1 \leq k \leq r$ with $\lambda_k \gneqq \lambda_{k+1}$. We want to prove that, for this special element, the set $\bar{e}((x),E_u)$ are all increasing chains of x. This property will be essential in §9.8 for us to make the D-function tables of pairs of special elements.

(1) The special elements in the sequence $\xi(w,k)$ will be introduced first in the case $k = r$, when they will be denoted by $x_1,\ldots,x_{\lambda_r}$, $x'_1,\ldots,x'_{\lambda_r}$. These elements are defined as follows.

$$\begin{cases} x_h = (\rho^{A_r^o}_{A_1,\ldots,A_{r-1},A_r^1})^{h-1}(w), & 1 < h < \lambda_r \\ x'_{h'} = \rho^{A_r^o}_{A_1,\ldots,A_{r-1}}(\rho^{A_r^o}_{A_1,\ldots,A_{r-1},A_r^1})^{h'-1}(w), & 1 < h' < \lambda_r \end{cases}$$

where $A_r^o(w)$ (resp. $A_r^1(w)$) is the block consisting of the first row (resp. the last (λ_r-1) rows) of $A_r(w)$. Then by the result in §9.2, all x_h, $1 < h < \lambda_r$, exist. Let us denote the u-th entry of $A_r^1(x_h)$ (resp. $A_r^o(x_h)$, $A_t(x_h)$, $1 < t < r$) by $e((x_h), j^u_{A_r^1}(x_h))$ (resp. $e((x_h), j^u_{A_r^o}(x_h))$, $e((x_h), j^u_t(x_h))$).

Then we have: for $1 < h < \lambda_r$,

$$\begin{cases} j_1^u(x_h),\ldots,j^u_{\mu_u}(x_h)) = (j_1^u(w),\ldots,j^u_{\mu_u}(w)), & \lambda_r < u < \lambda_1. \\ j_1^u(x_h),\ldots,j^u_{r-1}(x_h),j^{u-1}_{A_r^1}(x_h)) = (j_1^u(w),\ldots,j_r^u(w)), & h+1 < u < \lambda_r. \\ j_1^u(x_h),\ldots,j^u_{r-1}(x_h),j^u_{A_r^1}(x_h)) = (j_r^u(w)+n,j_1^u(w),\ldots,j^u_{r-1}(w)), & 1 < u < h-1. \\ j_1^h(x_h),\ldots,j^h_{r-1}(x_h),j^1_{A_r^o}(x_h)) = (j_1^h(w),\ldots,j_r^h(w)) & \end{cases} \quad (4)$$

and for $1 < h' < \lambda_r$,

$$\begin{cases} j_1^u(x'_{h'}),\ldots,j_{\mu_u}^u(x'_{h'})) = (j_1^u(w),\ldots,j_{\mu_u}^u(w)), & \lambda_r < u < \lambda_1. \\ j_1^u(x'_{h'}),\ldots,j_{r-1}^u(x'_{h'}),j_{A_r^1}^{u-1}(x'_{h'})) = (j_1^u(w),\ldots,j_r^u(w)), & h'+1 < u < \lambda_r. \\ j_1^u(x'_{h'}),\ldots,j_{r-1}^u(x'_{h'}),j_{A_r^1}^u(x'_{h'})) = (j_r^u(w)+n,j_1^u(w),\ldots,j_{r-1}^u(w)), & 1<u<h'-1. \\ j_1^{h'}(x'_{h'}),\ldots,j_{r-1}^{h'}(x'_{h'}),j_{A_r^o}^1(x'_{h'})) = (j_r^{h'}(w)+n,j_1^{h'}(w),\ldots,j_{r-1}^{h'}(w)) & . \end{cases} \tag{5}$$

Hence, from the fact that $\bar{e}((w),E_u)$, $1 < u < \lambda_1$, are all increasing chains of w, we see that $\bar{e}((x_h),E_u)$ (resp. $\bar{e}(x'_{h'}),E_u))$, $1 < u < \lambda_1$, are all increasing chains of x_h (resp. $x'_{h'}$) for any $1 < h$, $h' < \lambda_r$.

(2) We now define our special elements in the case when k satisfies $1 < k < r$ and $\alpha_{\lambda,0} = \lambda_k - \lambda_{k+1} > 0$. The elements in this case will be denoted by $y_1,y_2,\ldots,y_{\alpha_{\lambda,0}}$, $y'_o,y'_1,\ldots,y'_{\alpha_{\lambda,0}}$, z. They are defined as follows.

$$\begin{cases} y_h = \left(\rho^{A_k^o,A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1},A_k^1}\right)^{h-1} \rho^{A_r,\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1},A_k^1}(w), & 1 < h < \alpha_{\lambda,0} \\ y'_{h'} = \rho^{A_{i_1}^o}_{A_k^1}\,\rho^{A_{i_1}}_{A_1,\ldots,A_{k-1}}\,\rho^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}\left(\rho^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}\right)^{h'}(w), & 0 < h' < \alpha_{\lambda,0} \\ z = \rho^{A_r,\ldots,A_{k+1},A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}(w) \end{cases}$$

where $1 < k < r$ with $\lambda_k \gtrless \lambda_{k+1}$ and A_k^o, A_k^1, $A_{i_1}^o$, $A_{i_1}^1$, $\alpha_{\lambda,0}$, $i_1,\ldots,i_t$ are defined as before. Then it is clear that all y_h, $y'_{h'}$, $0 < h$, $h' < \alpha_{\lambda,0}$, z lie in $\xi(w,k)$.

(2a) We first investigate the detailed properties of y_1.

$y_1 \in H_{\lambda,k}$ has the semi-standard MDC form

$$([A_k^o, A_r], A_{r-1}, \ldots, A_{k+1}, A_k^1, A_{k-1}, \ldots, A_1) \text{ at } i', \quad i' = i + \sum_{v=k+1}^{r} \lambda_v,$$

where $H_{\lambda,k} = \eta^{-1}(\tilde{H}_{\lambda,k})$. Since w has local MDC form $(A_k^1, A_{k-1}, \ldots, A_1, A_r, \ldots, A_{k+1})$ at $i' + 1$ which is quasi-normal, by Lemma 5.4.5, for $1 < u < \alpha$, we have

$$(j_r^u(y_1), \ldots, j_{k+1}^u(y_1), j_{A_k^1}^u(y_1), j_{k-1}^u(y_1), \ldots, j_1^u(y_1))^{(om)}$$

$$= (j_{A_k^1}^u(w), j_{k-1}^u(w), \ldots, j_1^u(w), j_r^u(w) + n, \ldots, j_{k+1}^u(w) + n)^{(om)}$$

where
$$\alpha = \begin{cases} \lambda_1 & \text{if } k \neq 1 \\ \lambda_1 - 1 & \text{if } k = 1 \end{cases}$$

Let

$$\tilde{j}_h^u(y_1) = \begin{cases} j_h^{u-1}(y_1) & \text{for } {}^{(i)}\ h = i_v,\ 1 < v < t,\ \lambda_{i_{v+1}} + 2 < u < \lambda_{i_v} + 1 \\ & \qquad {}^{(ii)}\ h = A_k^1,\ \lambda_{i_1} + 2 < u < \lambda_k. \\ j_h^u(y_1) & \text{otherwise.} \end{cases}$$

Then

$$(\tilde{j}_{A_k^o}^u(y_1), \tilde{j}_r^u(y_1), \ldots, \tilde{j}_{k+1}^u(y_1), \tilde{j}_{A_k^1}^u(y_1), \tilde{j}_{k-1}^u(y_1), \ldots, \tilde{j}_1^u(y_1))^{(om)}$$

$$= (j_k^u(w), \ldots, j_1^u(w), j_r^u(w) + n, \ldots, j_{k+1}^u(w) + n)^{(om)} \text{ for } 1 < u < \lambda_1.$$

It follows that the sets $\bar{e}((y_1), E_u)$, $1 < u < \lambda_1$, are increasing chains of y_1.

If we observe y_1 more closely, we find that for any $1 < u < \lambda_1$, when $\lambda_{i_{m+1}} + 1 < u < \lambda_{i_m}$, $1 < m < t$, $\bar{e}((y_1),E_u)$ corresponds to the set of the $(u-\lambda_{i_{m+1}})$-th entries of all MDC blocks of $\tilde{y}_1^m$ under the map $d(\tau^{m+1},\tau^{m+1}_{i_{m+1}})\ldots d(\tau^t,\tau^t_{i_t})\cdot\eta$, where $\tilde{y}^m = d(\tau^{m+1},\tau^{m+1}_{i_{m+1}})\ldots d(\tau^t,\tau^t_{i_t})\cdot(\eta(y_1))$. When $\lambda_{i_1} < u < \lambda_1$, $\bar{e}((y_1),E_u)$ corresponds to the set of the $(u-\lambda_{i_1})$-th entries of all MDC blocks A_v, $1 < v < \mu_u$, of $\tilde{y}_1^o$, under the map $d(\tau^1,\tau^1_{i_1})\ldots d(\tau^t,\tau^t_{i_t})\cdot\eta$. This fact will be useful for the investigation in (2b), (2c) and (2d).

(2b) We now investigate the properties of y_h for $2 < h < \alpha_{\lambda,0}$. We have

$$y_h = (\rho^{[A_k^o,A_r],A_{r-1},\ldots,A_{k+1}}_{A_1,\ldots,A_{k-1},A_k^1})^{h-1}(y_1).$$

For $1 < m < t$, $\tilde{y}_1^m \in \tilde{H}_{\tau^m,k} \subseteq \tilde{F}(\tau^m,\tau^m_{i_m})$ has the semi-standard MDC form $([A_k^o,A_{i_m}],A_{i_m-1},\ldots,A_{k+1},A_k^1,A_{k-1},\ldots,A_1)$ which is normal for the first $\tau^m_{i_m}$ layers and $\tilde{y}_1^o \in \tilde{N}_{\tau^o}$ has the standard MDC form $([A_k^o,A_k^1],A_{k-1},\ldots,A_1)$, where for $0 < m < t$, $A_k^o(\tilde{y}_1^m)$ comes from $[A_k^o(\tilde{y}_1^{m+1}),A_{i_{m+1}}(\tilde{y}_1^{m+1}),\ldots,A_{i_m+1}(\tilde{y}_1^{m+1})]$ and $A(\tilde{y}_1^m)$ comes from $A(\tilde{y}_1^{m+1})$ for $A \in \{A_{i_m},\ldots,A_{k+1},A_k^1,A_{k-1},\ldots,A_1\}$.

By Lemma 8.5.13, for $2 < h < \alpha_{\lambda,0}$, $\tilde{y}_h^t = \eta(y_h)$ has full MDC form $([A_k^o,A_r],\ A_{r-1},\ldots,A_{k+1},\ A_k^1,\ A_{k-1},\ldots,A_1)$ which is normal for the first $\tau^t_{i_t}$ layers. So there exists $\tilde{y}_h^{t-1} = d(\tau^t,\tau^t_{i_t})\cdot(\eta(y_h))$.

By Corollary 8.8.2 and Lemma 8.9.1,

$$\tilde{y}_h^{t-1} = \left(\wp^{[A_k^o, A_{i_{t-1}}], A_{i_{t-1}-1}, \ldots, A_{k+1}}_{A_1, \ldots, A_{k-1}, A_k^1}\right)^{h-1} (\tilde{y}_1^{t-1}).$$

If $t > 1$, then by Lemma 8.5.13, $\tilde{y}_h^{t-1}$ has full MDC form $([A_k^o, A_{i_{t-1}}], A_{i_{t-1}-1}, \ldots, A_{k+1}, A_k^1, A_{k-1}, \ldots, A_1)$ which is normal for the first $\tau_{i_{t-1}}^{t-1}$ layers. By applying induction on $t-m > 0$, this implies that for $2 < h < \alpha_{\lambda,0}$, there exists

$$\tilde{y}_h^m = d(\tau^{m+1}, \tau_{i_{m+1}}^{m+1}) \ldots d(\tau^t, \tau_{i_t}^t)(\eta(y_h)) = \left(\wp^{[A_k^o, A_{i_m}], A_{i_m-1}, \ldots, A_{k+1}}_{A_1, \ldots, A_{k-1}, A_k^1}\right)^{h-1} (\tilde{y}_1^m)$$

having full MDC form $([A_k^o, A_{i_m}], A_{i_m-1}, \ldots, A_{k+1}, A_k^1, A_{k-1}, \ldots, A_1)$ which is normal for the first $\tau_{i_m}^m$ layers for $1 < m < t$, and

$$\tilde{y}_h^o = d(\tau^1, \tau_{i_1}^1) \ldots d(\tau^t, \tau_{i_t}^t)(\eta(y_h)) = \left(\wp^{A_k^o}_{A_1, \ldots, A_{k-1}, A_k^1}\right)^{h-1} (\tilde{y}_1^o) \in \tilde{H}^h_{\tau^o, k}$$

having full MDC form $(A_k^o, A_k^1, A_{k-1}, \ldots, A_1)$, where $\tilde{H}^h_{\tau^o,k} = \eta(H^h_{\tau^o,k})$.

So for any u with $\lambda_{i_{m+1}}+1 < u < \lambda_{i_m}$, $1 < m < t$, the set $\bar{e}((\tilde{y}_h^m), E_{u-\lambda_{i_{m+1}}})$ of the $(u-\lambda_{i_{m+1}})$-th entries of all MDC blocks of $\tilde{y}_h^m$ comes from the set $\bar{e}((\tilde{y}_1^m), E_{u-\lambda_{i_{m+1}}})$ of the $(u-\lambda_{i_{m+1}})$-th entries of all MDC blocks of $\tilde{y}_1^m$ under the operation $\left(\wp^{[A_k^o, A_{i_m}], A_{i_m-1}, \ldots, A_{k+1}}_{A_1, \ldots, A_{k-1}, A_k^1}\right)^{h-1}$, and $\bar{e}((\tilde{y}_1^m), E_{u-\lambda_{i_{m+1}}})$ comes from $\bar{e}((\tilde{y}_1), E_u)$ under the map $d(\tau^{m+1}, \tau_{i_{m+1}}^{m+1}) \ldots d(\tau^t, \tau_{i_t}^t) \cdot \eta$. This implies by Corollary 8.8.2 and Lemma 8.9.1 that $\bar{e}((\tilde{y}_h^m), E_{u-\lambda_{i_{m+1}}})$ comes from $\bar{e}((y_h), E_u)$. Since $\bar{e}((\tilde{y}_h^m), E_{u-\lambda_{i_{m+1}}})$ is an increasing chain of $\tilde{y}_h^m$, it follows that $\bar{e}((y_h), E_u)$ is an increasing chain of y_h, too.

For any u with $\lambda_{i_1}+1 < u < \lambda_1$, the set $\bar{e}((\tilde{y}_1^o), E_{u-\lambda_{i_1}})$ of the $(u-\lambda_{i_1})$-th entries of all MDC blocks of $\tilde{y}_1^o$ comes from $\bar{e}((y_1), E_u)$ under the map $d(\tau^1, \tau^1_{i_1}) \ldots d(\tau^t, \tau^t_{i_t}) \cdot \eta$. Assume that $\bar{e}(\tilde{y}_h^o)$ is the entry set of $\tilde{y}_h^o$ coming from $\bar{e}((\tilde{y}_1^o), E_{u-\lambda_{i_1}})$ under the operation $(\wp^{A_k^o}_{A_1,\ldots,A_{k-1},A_k^1})^{h-1}$. Then by Corollary 8.8.2 and Lemma 8.9.1, $\bar{e}(\tilde{y}_h^o)$ comes from $\bar{e}((y_h), E_u)$. By equation (4), $\bar{e}(\tilde{y}_h^o)$ is an increasing chain of $\tilde{y}_h^o$. So $\bar{e}((y_h), E_u)$ is an increasing chain of y_h, too. Therefore for all u,h with $1 < u < \lambda_1$ and $1 < h < \alpha_{\lambda,0}$, $\bar{e}((y_h), E_u)$ is an increasing chain of y_h.

(2c) We next investigate the properties of y_h' for all h. Let us rewrite

$$\begin{cases} y_o' = \wp^{A^o_{i_1}}_{A_k^1} \wp^{A_{i_1}}_{A_1,\ldots,A_{k-1}} \wp^{A_r,\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}(w) \\ y_h' = \wp^{A^o_{i_1}}_{A_k^1} \wp^{A_{i_1}}_{A_1,\ldots,A_{k-1}} \wp^{[A_k^o,A_r],A_{r-1},\ldots,A_{i_1+1}}_{A_1,\ldots,A_{k-1},A_k^1}(y_h), \quad 1 < h < \alpha_{\lambda,0} \end{cases}$$

Clearly, y_h' belongs to the sequence $\xi(w,k)$ beginning with w and has full MDC form

$$(A_{i_1-1},\dots,A_{k+1},[A_k^o,A_r],A_{r-1},\dots,A_{i_1+1},A_{i_1}^o,A_k^1,A_{i_1}^1,A_{k-1},\dots,A_1)$$

at $i + \sum_{v=i_1}^{r} \lambda_v + h\cdot(\sum_{v=k+1}^{r} \lambda_v + 1)$ for $0 < h < \alpha_{\lambda,0}$, and $y'_{\alpha_{\lambda,0}}$ has full MDC form $(A_{i_1-1},\dots,A_{k+1},[A_k^o,A_r],A_{r-1},\dots,A_{i_1+1},A_{i_1}^o,[A_k^1,A_{i_1}^1],$ $A_{k-1},\dots,A_1)$ at $i + \sum_{v=i_1}^{r} \lambda_v + \alpha_{\lambda,0}\cdot(\sum_{v=k+1}^{r} \lambda_v + 1)$ since $\rho_{A_k^1}^{A_{i_1}^1}(y'_h)$ exists for any $0 < h < \alpha_{\lambda,0}$ but $\rho_{A_k^1}^{A_{i_1}^1}(y'_{\alpha_{\lambda,0}})$ doesn't.

Let $\tilde{y}_h^{\prime t} = \eta(y'_h)$, $0 < h < \alpha_{\lambda,0}$. Let $\tilde{y}_h^{\prime m} \in \tilde{\mathcal{A}}_{\beta_{\lambda,m}}$ be obtained from $\tilde{y}_h^{\prime m+1}$ by deleting the first $\tau_{i_{m+1}}^{m+1}$ entries of the MDC blocks $A_{i_1-1}(\tilde{y}_h^{\prime m+1}),\dots,A_{k+1}(\tilde{y}_h^{\prime m+1})$, $[A_k^o(\tilde{y}_h^{\prime m+1}),A_{i_{m+1}}(\tilde{y}_h^{\prime m+1})]$, $A_{i_{m+1}-1}(\tilde{y}_h^{\prime m+1}),\dots,A_{i_1+1}(\tilde{y}_h^{\prime m+1}),A_{i_1}^o(\tilde{y}_h^{\prime m+1})$, $A_k^1(\tilde{y}_h^{\prime m+1})$, $A_{k-1}(\tilde{y}_h^{\prime m+1}),\dots,A_1(\tilde{y}_h^{\prime m+1})$, $0 < m < t$. Then $\tilde{y}_h^{\prime m}$ has full MDC form

$$\begin{cases} (A_{i_1-1},\dots,A_{k+1},[A_k^o,A_{i_m}],A_{i_m-1},\dots,A_{i_1+1},A_{i_1}^o,A_k^1,A_{i_1}^1,A_{k-1},\dots,A_1), & m > 1. \\ (A_{i_1-1},\dots,A_{k+1},\ [A_k^o,A_{i_1}^o],\ A_k^1,\ A_{i_1}^1,\ A_{k-1},\dots,A_1), & m = 1. \\ (A_k^1,\ A_{i_1}^1,\ A_{k-1},\dots,A_1), & m = 0 \end{cases}$$

$\tilde{y}'^{m}_{\alpha_{\lambda,0}}$ has full MDC form

$$\begin{cases} (A_{i_1-1},\ldots,A_{k+1},[A_k^o,A_{i_m}],\ A_{i_m-1},\ldots,A_{i_1+1},A_{i_1}^o,[A_k^1,A_{i_1}^1],A_{k-1},\ldots,A_1), & m > 1. \\ (A_{i_1-1},\ldots,A_{k+1},[A_k^o,A_{i_1}^o],\ [A_k^1,A_{i_1}^1],A_{k-1},\ldots,A_1), & m = 1. \\ ([A_k^1,A_{i_1}^1],A_{k-1},\ldots,A_1), & m = 0 \end{cases}$$

where $A_k^o(\tilde{y}_h^{\prime m})$ comes from $[A_k^o(\tilde{y}_h^{\prime m+1}),A_{i_{m+1}}(\tilde{y}'^{m+1}_h),\ldots,A_{i_m+1}(\tilde{y}_h^{\prime m+1})]$ for $m > 1$ and $A(\tilde{y}_h^{\prime m})$ comes from $A(\tilde{y}_h^{\prime m+1})$ for $A \neq A_k^o$ under the deletion operation. Clearly, for any MDC blocks $A(\tilde{y}_h^{\prime m})$ of $\tilde{y}_h^{\prime m}$ with $A \neq A_{i_1}^1$, we have $|A(\tilde{y}_h^{\prime m})| > \lambda_{i_m} - \lambda_{i_{m+1}}$ when $1 < m < t$.

By Corollary 8.8.2, and Lemma 8.9.1, we can easily show that

$$\begin{cases} \tilde{y}_o^{\prime m} = \wp_{A_k^1}^{A_{i_1}^o} \wp_{A_1,\ldots,A_{k-1}}^{A_{i_1}} \wp_{A_1,\ldots,A_{k-1},A_k^1}^{A_{i_m},\ldots,A_{i_1+1}}(\tilde{y}_o^m) \\ \tilde{y}_h^{\prime m} = \wp_{A_k^1}^{A_{i_1}^o} \wp_{A_1,\ldots,A_{k-1}}^{A_{i_1}} \wp_{A_1,\ldots,A_{k-1},A_k^1}^{[A_k^o,A_{i_m}],A_{i_m-1},\ldots,A_{i_1+1}}(\tilde{y}_h^m), \quad 1 < h < \alpha_{\lambda,0} \end{cases}$$

for $2 < m < t$, and

$$\begin{cases} \tilde{y}_o^{\prime 1} = \wp_{A_k^1}^{A_{i_1}^o} \wp_{A_1,\ldots,A_{k-1}}^{A_{i_1}}(\tilde{y}_o^1) \\ \tilde{y}_h^{\prime 1} = \wp_{A_k^1}^{A_{i_1}^o} \wp_{A_1,\ldots,A_{k-1}}^{A_{i_1}} \wp_{A_1,\ldots,A_{k-1},A_k^1}^{A_k^o}(\tilde{y}_h^1), \quad 1 < h < \alpha_{\lambda,0} \end{cases}$$

$$\tilde{y}_h^{\prime o} = \wp_{A_1,\ldots,A_{k-1}}^{A_{i_1}^1}(\tilde{y}_h^o), \quad 0 < h < \alpha_{\lambda,0}.$$

First, let us consider y_h', $1 < h < \alpha_{\lambda,0}$.

For any $1 \le u \le \lambda_{i_1}$, there exists m, $1 \le m \le t$, such that $\lambda_{i_{m+1}} + 1 \le u \le \lambda_{i_m}$. Let $\bar{e}(\tilde{y}_h'^m)$ be the entry set of $\tilde{y}_h'^m$ consisting of the $(u-\lambda_{i_{m+1}})$-th entries of all MDC blocks except $A^1_{i_1}$. Let $\bar{e}(\tilde{y}^m_h)$ be the entry set of $\tilde{y}^m_h$ consisting of the $(u-\lambda_{i_{m+1}})$-th entries of all MDC blocks. Then since $\tilde{y}^m_h$ has full MDC form $([A^0_k, A_{i_m}], A_{i_m-1}, \ldots, A_{k+1}, A^1_k, A_{k-1}, \ldots, A_1)$ which is normal for the first $\lambda_{i_m} - \lambda_{i_{m+1}}$ layers, by Lemma 8.5.12, $\bar{e}(\tilde{y}_h'^m)$ comes from $\bar{e}(\tilde{y}^m_h)$ under the ρ operations from $\tilde{y}^m_h$ to $\tilde{y}_h'^m$ and so is an increasing chain of $\tilde{y}_h'^m$. Since $\bar{e}(\tilde{y}^m_h)$ comes from $\bar{e}(y_h), E_u)$ under the map $d(\tau^{m+1}, \tau^{m+1}_{i_{m+1}}) \ldots d(\tau^t, \tau^t_{i_t}) \cdot \eta$, by Corollary 8.8.2 and Lemma 8.9.1, $\bar{e}(\tilde{y}_h'^m)$ comes from $\bar{e}((y_h'), E_u)$ under the map η and the deleting operation. So $\bar{e}((y_h'), E_u)$ is an increasing chain of y_h'.

For any u with $\lambda_{i_1} + 1 \le u \le \lambda_1$, let $\bar{e}(\tilde{y}_h'^0)$, $0 \le h \le \alpha_{\lambda,0}$, be the entry set of $\tilde{y}_h'^0$ coming from the entry set $\bar{e}(\tilde{y}^0_1)$ under the operation

$$\rho^{A^1_{i_1}}_{A_1,\ldots,A_{k-1}} \left(\rho^{A^1_{i_1}}_{A_1,\ldots,A_{k-1},A^1_k}\right)^{h-1},$$

where $\bar{e}(\tilde{y}^0_1)$ is the entry set of $\tilde{y}^0_1$ consisting of the $(u-\lambda_{i_1})$-th entries of all MDC blocks. Then by Equation (5) and noting that $A^1_{i_1}$ plays the role of A^0_k in this equation, we see that $\bar{e}(\tilde{y}_h'^0)$ is an increasing chain of $\tilde{y}_h'^0$. Since $\bar{e}(\tilde{y}^0_1)$ comes from $\bar{e}((y_1), E_u)$ under the map $d(\tau^1, \tau^1_{i_1}) \ldots d(\tau^t, \tau^t_{i_t}) \cdot \eta$, by Corollary 8.8.2 and

Lemma 8.9.1, $\bar{e}(\tilde{y}_h'^{o})$ comes from $\bar{e}((y_h'),E_u)$ under the deleting operation. So $\bar{e}((y_h'),E_u)$ is also an increasing chain of y_h'.

Secondly, we consider y_o' which has full MDC form

$$(A_{i_1-1},\ldots,A_{k+1},[A_k^o,A_r],A_{r-1},\ldots,A_{i_1+1},A_{i_1}^o,A_k^1,A_{i_1}^1,A_{k-1},\ldots,A_1)$$

at $i + \sum_{v=i_1}^{r} \lambda_v$. Now w has full local MDC form

$(A_k^1,A_{k-1},\ldots,A_1,A_r,\ldots,A_{i_1})$ at $i + 1 + \sum_{v=k+1}^{r} \lambda_v$ which is quasi-normal. By Lemma 5.4.5, we have

$$(j_r^u(y_o'),\ldots,j_{i_1}^u(y_o'),j_{A_k^1}^u(y_o'),\ j_{k-1}^u(y_o'),\ldots,j_1^u(y_o'))^{(om)}$$

$$= (j_{A_k^1}^u(w),j_{k-1}^u(w),\ldots,j_1^u(w),j_r^u(w)+n,\ldots,j_{i_1}^u(w)+n)^{(om)} \quad \text{for } 1 < u < \lambda_1.$$

So if we let

$$\tilde{j}_v^u(y_o') = \begin{cases} j_v^{u-1}(y_o') & \text{for (i) } v = i_m,\ 2 < m < t,\ \lambda_{i_{m+1}} + 2 < u < \lambda_{i_m} + 1 \\ & \text{(ii) } v = A_{i_1}^o,\ \lambda_{i_2} + 2 < u < \lambda_{i_1} \\ & \text{(iii) } v = A_k^1,\ \lambda_{i_1} + 1 < u < \lambda_k \\ j_{A_{i_1}^1}^{u-\lambda_{i_1}+1}(y_o') & \text{for } v = A_{i_1}^1 \\ j_v^u(y_o') & \text{otherwise} \end{cases}$$

then

$$(\tilde{j}_{i_1-1}^u(y_o'),\ldots,\tilde{j}_{k+1}^u(y_o'),\tilde{j}_{A_k^o}^u(y_o'),\tilde{j}_r^u(y_o'),\ldots,\tilde{j}_{i_1+1}^u(y_o'),\tilde{j}_{A_{i_1}^o}^u(y_o'),\tilde{j}_{A_k^1}^u(y_o'),(\tilde{j}_{A_{i_1}^1}^u(y_o'),$$
$$\tilde{j}_{k-1}^u(y_o'),\ldots,\tilde{j}_1^u(y_o')))^{(om)} = (j_{i_1-1}^u(w),\ldots,j_1^u(w),j_r^u(w)+n,\ldots,j_{i_1}^u(w)+n)^{(om)} \quad \text{for}$$

$1 \le u \le \lambda_1$

This implies that $\bar{e}((y_o'),E_u)$, $1 \leq u \leq \lambda_1$, are all increasing chains of y_o'.

(2d) Finally we investigate the element z. Let us rewrite $z = \rho^{A_k^o}_{A_1,\ldots,A_{k-1},A_k^1}(y_1)$ which has full MDC form $(A_r,\ldots,A_{k+1},A_k^o,A_k^1,A_{k-1},\ldots,A_1)$ at $i + \sum_{v=k+1}^{r} \lambda_v + 1$. Since y_1 has MDC form $(A_k^1,A_{k-1},\ldots,A_1,A_k^o)$ at $i + 2\sum_{v=k+1}^{r} \lambda_v + 1$ which is normal for the first layer, it follows that

$$(j^1_{r-1}(z),\ldots,j^1_{k+1}(z),j^1_{A_k^o}(z),\ j^1_{A_k^1}(z),j^1_{k-1}(z),\ldots,j^1_1(z))$$

$$= (j^1_{r-1}(y_1),\ldots,j^1_{k+1}(y_1),j^1_{A_k^1}(y_1),j^1_{k-1}(y_1),\ldots,j^1_1(y_1),j^1_{A_k^o}(y_1) + n).$$

So $\bar{e}((z),E_1)$ is an increasing chain of z since $\bar{e}((y_1),E_1)$ is an increasing chain of y_1. For $2 \leq u \leq \lambda_1$, it is easily seen from the corresponding assertion on $\bar{e}((y_1),E_u)$ that $\bar{e}((z),E_u)$ is an increasing chain of z.

By combining (1), (2a), (2b), (2c) and (2d), we have now proved the main result in this section, which is as follows.

<u>Proposition 9.7.2</u> <u>Let $w \in N_\lambda$ and x_h, x_h', $y_{h'}$, $y_{h'}'$, z in $\xi(w,k)$ and $E_u \subseteq \underline{n}$ be defined as above with $1 \leq k \leq r$, $0 \leq h \leq \lambda_r$, $0 \leq h' \leq \alpha_{\lambda,0}$ and $1 \leq u \leq \lambda_1$. Then for any $1 \leq u \leq \lambda_1$, $\bar{e}((x),E_u)$ is an increasing chain of x for $x = x_h$, x_h', $y_{h'}$, $y_{h'}'$, z.</u> □

As a by-product of the above discussion, we also have:

Lemma 9.7.3 Let $w \in N_\lambda$ and $x_h, x'_h, y_{h'}, y'_{h'}, z$ in $\xi(w,k)$ and $E_u \subseteq \underline{n}$ be defined as in Lemma 9.7.2 with $1 \leq k \leq r$, $0 \leq h \leq \lambda_r$, $0 \leq h' \leq \alpha_{\lambda,0}$ and $1 \leq u \leq \lambda_1$. For any MDC block $A(x)$ of x with $x = x_h, x'_h, y_{h'}, y'_{h'}, z$, let $e((x), j_A^\alpha(x))$, $e((x), j_A^\beta(x))$ be the α-th and the β-th entry of $A(x)$, respectively. Suppose $e((x), j_A^\alpha(x)) \in \bar{e}((x), E_u)$ and $e((x), j_A^\beta(x)) \in \bar{e}((x), E_v)$. Then $\alpha > \beta$ implies $u > v$. □

§9.8 THE D-FUNCTION

We shall in this section define a D-function which will be used in Chapter 10 to show that each raising operator on layers transforms an element w into an element w' in the same left cell as w, provided that both w and w' lie in N_λ.

Definition 9.8.1 For any $E \subseteq \underline{n}$, we define a map d_E: $\mathbb{Z} \times \mathbb{Z} \to \frac{1}{2}\mathbb{Z}$ as follows. For $\beta_1, \beta_2 \in \mathbb{Z}$ with $\beta_1 \leq \beta_2$, let $B_{\beta_1,\beta_2} = \{a \in \mathbb{Z} \mid \beta_1 \leq a \leq \beta_2,\ \bar{a} \in E\}$. Then we define

$$d_E(\beta_1,\beta_2) = -d_E(\beta_2,\beta_1) = |B_{\beta_1,\beta_2}| - \frac{1}{2}\,|\{\bar{\beta}_1, \bar{\beta}_2\} \cap E|.$$

Since $d_E(\beta,\beta) = 0$ for any $\beta \in \mathbb{Z}$, the map d_E is well defined. It is easily seen that for any $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$, we have

$$d_E(\beta_1,\beta_3) = d_E(\beta_1,\beta_2) + d_E(\beta_2,\beta_3) \qquad (6)$$

For any $\emptyset \neq E \subseteq \underline{n}$, let $z_h \in A_n$, $h = 1,2$, have a DC form (A_h) at α_h. Assume that $\bar{e}((z_h),E)$ is an increasing chain of z_h, then there exists at most one entry of A_h lying in $\bar{e}((z_h),E)$ by Lemma 3.10.

Assume that $\alpha_1 < \alpha_2$ and $|A_h| = m_h$, $h = 1,2$. Let $e_{z_1}(\beta_1,\gamma_1)$ be the entry of z_1 lying in $\bar{e}((z_1),E)$ with $\beta_1 > \alpha_1$ and β_1 as small as possible. Let $e_{z_2}(\beta_2,\gamma_2)$ be the entry of z_2 lying in $\bar{e}((z_2),E)$ with $\beta_2 < \alpha_2 + m_2$ and β_2 as large as possible. Let

$$E^{z_2,A_2}_{z_1,A_1}(E) = \begin{cases} 0 & \text{if } e_{z_1}(\beta_1,\gamma_1) \text{ lies in } A_1 \text{ and } e_{z_2}(\beta_2,\gamma_2) \text{ in } A_2. \\ 2 & \text{if neither } e_{z_1}(\beta_1,\gamma_1) \text{ lies in } A_1 \text{ nor } e_{z_2}(\beta_2,\gamma_2) \text{ in } A_2. \\ 1 & \text{otherwise.} \end{cases}$$

We define

$$D^{z_2,A_2}_{z_1,A_1}(E) = -D^{z_1,A_1}_{z_2,A_2}(E) = d_E(\gamma_1,\gamma_2) + \frac{1}{2}\, E^{z_2,A_2}_{z_1,A_1}(E)$$

and say that the layer $\bar{e}(E)$ is raised in $D^{z_2,A_2}_{z_1,A_1}(E)$ terms from (z_1,A_1) to (z_2,A_2). (Note that $D^{z_2,A_2}_{z_1,A_1}(E)$ is a half-integer - it need not be an integer in general).

Assume that z_3 has a DC form (A_3) at α_3 with $\bar{e}((z_3),E)$ being an increasing chain of z_3.

<u>Lemma 9.8.2</u>. <u>Let the elements $z_h \in A_n$, the blocks A_h of z_h, $1 < h < 3$, and the set of residue classes $E \subseteq \underline{n}$ be as above. Then</u>

$$D^{z_3,A_3}_{z_1,A_1}(E) = D^{z_2,A_2}_{z_1,A_1}(E) + D^{z_3,A_3}_{z_2,A_2}(E)$$

Proof By definition of D-function and formula (6). □

Lemma 9.8.3 Assume that w, w' have full DC forms $(A_\ell,\ldots,A_1)$, $(A'_\ell,\ldots,A'_1)$ at i, i', respectively, satisfying the following conditions.

(i) There exists a decomposition $\underline{n} = E_1 \cup \ldots \cup E_t$ such that $\bar{e}((x),E_u)$ is an increasing chain of x for all $x = w, w'$, $1 \leq u \leq t$.

(ii) $|A_j| = |A'_j|$, for $1 \leq j \leq \ell$.

(iii) For any $1 \leq j \leq \ell$, $1 \leq u \leq t$, $D^{w',A'_j}_{w\;\;,A_j}(E_u) = 0$.

Then $w = w'$.

Proof By (ii), (iii), the set of integers labelling the columns of w containing some entry of A_t is the same as the set of integers labelling the columns of w' containing some entry of A'_t for any $1 \leq t \leq \ell$.

Let $e((w),j^u_t(w))$ (resp. $e((w'),j^u_t(w'))$ be the u-th entry of $A_t(w)$ (resp. $A'_t(w')$). Since A_t, A'_t are both DC blocks, this implies $j^u_t(w) = j^u_t(w')$ for $1 \leq u \leq |A_t|$. So

$\sum_{\substack{1\leq t\leq \ell \\ 1\leq u\leq |A_t|}} j^u_t(w) = \sum_{\substack{1\leq t\leq \ell \\ 1\leq u\leq |A'_t|}} j^u_t(w')$. Since $(A_\ell,\ldots,A_1)$, $(A'_\ell,\ldots,A'_1)$ are full DC forms of w, w', respectively, by definition of affine matrix, we have $i = i'$. Hence $w = w'$. □

Corollary 9.8.4 Assume that w, w', w'' have full DC forms $(A_\ell,\dots,A_1)$, $(A'_{i_\ell},\dots,A'_{i_1})$, $(A''_{i_\ell},\dots,A''_{i_1})$ at i, i', i'', respectively. Suppose they also satisfy the following conditions.

(i) There exists a decomposition $\underline{n} = E_1 \cup \dots \cup E_t$ such that $\bar{e}((x),E_u)$ is an increasing chain of x for any $x \in \{w,w',w''\}$, $1 < u < t$.

(ii) There exists a permutation $s \in S_\ell$ such that

$(s(\ell),\dots,s(1)) = (i_\ell,\dots,i_1)$ and $|A_j| = |A'_{s(j)}| = |A''_{s(j)}|$, $1 < j < \ell$.

(iii) For any $1 < j < \ell$, $1 < u < t$, we have

$$D^{w',A'_{s(j)}}_{w,A_j}(E_u) = D^{w'',A''_{s(j)}}_{w,A_j}(E_u).$$ Then $w' = w''$.

Proof By Lemma 9.8.2, we see that $D^{w'',A''_{s(j)}}_{w',A'_{s(j)}}(E_u) = 0$ for any $1 < j < \ell$, $1 < u < t$. So our result follows from Lemma 9.8.3. □

Example 9.8.5

(i) Let $z_1,z_2 \in A_n$ be such that z_1 has full MDC form $(A_\ell,\dots,A_1)$ at $i \in \mathbb{Z}$ which is normal for the first m layers with $|A_1| > \dots > |A_\ell|$ and $z_2 = \rho^{A_\ell,\dots,A_{v+1}}_{A_1,\dots,A_v}(z_1)$ for $1 < v < \ell$. Let

$E_u = \overline{\{j^u_h(z_1)}\,|\,1 < h < \alpha_u\}$, $1 < u < m$, $\alpha_u = \max\,\{h\,|\,1 < h < \ell, |A_h| > u\}$.

Then $\bar{e}(E_u)$ is raised in $D^{z_2,A_t(z_2)}_{z_1,A_t(z_1)}(E_u)$ terms from $(z_1,A_t(z_1))$ to

$(z_2, A_t(z_2))$, for any t, u with $1 < t < \ell$ and $1 < u < m$, where

$$D^{z_2,A_t(z_2)}_{z_1,A_t(z_1)}(E_u) = \begin{cases} \alpha_u - v & \text{if } \alpha_u > v \\ 0 & \text{if } \alpha_u < v \end{cases}$$

(ii) Let $w \in N_\lambda$ have a standard MDC form $(A_r,\dots,A_1)$ at $i \in \mathbb{Z}$. Let $x_1 = w$, $x_2,\dots,x_{\lambda_r}$, $x'_1,\dots,x'_{\lambda_r}$ be such that for every $1 < j < \lambda_r$, $1 < j' < \lambda_r$, $x_j = \rho^{A_r^0}_{A_1,\dots,A_{r-1},A_r^1}(x_{j-1})$, $x'_{j'} = \rho^{A_r^0}_{A_1,\dots,A_{r-1}}(x_{j'})$, where A_r^0 (resp. A_r^1) is the block consisting of the first row (resp. the last $\lambda_r - 1$ rows) of A_r. Let $E_u = \{\overline{j^u_h(w)} \mid 1 < h < \mu_u\}$, $1 < u < \lambda_1$. Then by Proposition 9.7.2 and the formulae of ρ-operations, we can calculate the D-functions as follows:

Table (I)

$D^{x_j, A(x_j)}_{x_{j-1}, A(x_{j-1})}(E_u)$ — u \ A	$A_1, \ldots, A_{r-1}$	A_r^1	A_r^0
$1 < u < \lambda_1$, $u \neq j-1, j$	0	0	0
$j-1$	1	$\frac{1}{2}$	$\frac{1}{2}$
j	0	$\frac{1}{2}$	$\frac{1}{2}$

$1 < j < \lambda_r$

$D^{x'_j, A(x'_j)}_{x_j, A(x_j)}(E_u)$ — u \ A	$A_1, \ldots, A_{r-1}$	A_r^1	A_r^0
$1 < u < \lambda_r$, $u \neq j$	0	0	1
j	1	0	1
$\lambda_r < u < \lambda_1$	0	0	0

$1 < j < \lambda_r$

Table (II)

$D^{x_j, A(x_j)}_{w, A(w)}(E_u)$ — u \ A	$A_1, \ldots, A_{r-1}$	A_r^1	A_r^0
1	1	$\frac{1}{2}$	$\frac{1}{2}$
$2 < u < j-1$	1	1	1
j	0	$\frac{1}{2}$	$\frac{1}{2}$
$j < u < \lambda_r$	0	0	0
$\lambda_r < u < \lambda_1$	0	0	0

$1 < j < \lambda_r$

$D^{x'_j, A(x'_j)}_{w, A(w)}(E_u)$ — u \ A	$A_1, \ldots, A_{r-1}$	A_r^1	A_r^0
1	1	$\frac{1}{2}$	$\frac{3}{2}$
$2 < u < j-1$	1	1	2
j	1	$\frac{1}{2}$	$\frac{3}{2}$
$j < u < \lambda_r$	0	0	1
$\lambda_r < u < \lambda_1$	0	0	0

$1 < j < \lambda_r$

(iii) Let $w \in \mathbb{N}_\lambda$ have the standard MDC form $(A_r,\dots,A_1)$ at $i \in \mathbb{Z}$. Let $y_0 = w, y_1,\dots,y_{\alpha_{\lambda,0}}, y'_1,\dots,y'_{\alpha_{\lambda,0}}$ with $\alpha_{\lambda,0} = \lambda_k - \lambda_{k+1} > 0$ and $k < r$ such that

$$y_h = \left(\rho^{A^0_k, A_r,\dots,A_{k+1}}_{A_1,\dots,A_{k-1},A^1_k}\right)^{h-1} \rho^{A_r,\dots,A_{k+1}}_{A_1,\dots,A_{k-1},A^1_k}(w), \quad 1 < h < \alpha_{\lambda,0}$$

$$y'_{h'} = \rho^{A^0_{i_1}}_{A^1_k} \rho^{A_{i_1}}_{A_1,\dots,A_{k-1}} \rho^{A_r,\dots,A_{i_1+1}}_{A_1,\dots,A_{k-1},A^1_k} \left(\rho^{A_r,\dots,A_{k+1},A^0_k}_{A_1,\dots,A_{k-1},A^1_k}\right)^{h'}(w), \quad 0<h'<\alpha_{\lambda,0}.$$

where A^0_k (resp. A^1_k) is the block consisting of the first row (resp. the last $\lambda_k - 1$ rows) of A_k, $A^0_{i_1}$ (resp. $A^1_{i_1}$) is the block consisting of the first $\lambda_{i_1}-1$ rows (resp. the last row) of A_{i_1}.

Let $E_v = \{\overline{j^v_h(w)} \mid 1 < h < \mu_v\}$, $1 < v < \lambda_1$. Then by Proposition 9.7.2 and the formulae of ρ-operations, we can calculate the D-functions as follows, where the notation δ_{ij} in these tables is a Kronecker delta.

TABLE (III)

$D^{y_1,\Lambda(y_1)}_{w,\Lambda(w)}(E_v)$ (v \ Λ)	Λ_k^o	Λ_{i_g} $2 \le g \le t$	$\Lambda_{i_{g-1}+1},\ldots,\Lambda_{i_g-1}$ $2 \le g \le t$	$\Lambda_{i_1}^o$	$\Lambda_{i_1}^1$	$\Lambda_{i_o+1},\ldots,\Lambda_{i_1-1}$	Λ_k^1	Λ_a $1 \le a < k$
1	0	$(i_t-i_o)+\frac{1}{2}\delta_{gt}$	i_t-i_o	$(i_t-i_o)+\frac{1}{2}\delta_{t,1}$	$(i_t-i_o)+\frac{1}{2}\delta_{t1}\cdot\delta_{a_{\lambda,1}1}$	i_t-i_o	$i_t-i_o-\frac{1}{2}$	i_t-i_o
$\lambda_{i_h}+1$ $3 \le h \le t$	0	$(i_{h-1}-i_o)+\frac{1}{2}(\delta_{g,h}+\delta_{g,h-1})$	$(i_{h-1}-i_o)+\delta_{gh}$	$i_{h-1}-i_o$				
$\lambda_{i_h}+1 < v \le \lambda_{i_{h-1}}$ $1 \le h \le t+1$ $v \ne \lambda_{i_1}$	0	$i_{h-1} - i_o$						
$\lambda_{i_2}+1$ $v \ne 1$	0	$(i_1-i_o)+\frac{1}{2}\delta_{g,2}$	$(i_1-i_o)+\delta_{g,2}$	$(i_1-i_o)+\frac{1}{2}(1+\delta_{a_{\lambda,1},1})$	$(i_1-i_o)+\frac{1}{2}\delta_{a_{\lambda,1},1}$	i_1-i_o		
λ_{i_1} $v \ne \lambda_{i_2}+1$	0	$i_1 - i_o$		$(i_1-i_o)+\frac{1}{2}$	$(i_1-i_o)+\frac{1}{2}$	i_1-i_o		
$\lambda_{i_1}+1$	0				$\frac{1}{2}$	1	$\frac{1}{2}$	0
$\lambda_{i_o} < v \le \lambda_1$	0							

TABLE (III)

$1 < j \le a_{\lambda,0}$

D \ A: $y_j, A(y_j)$ (E_v) $y_{j-1}, A(y_{j-1})$ / v	$A^o_{k'}$,	A_a $1 \le a < k$ or $i_1 < a \le r$	$A^o_{i_1}$	$A^1_{i_1}$	A^1_k	A_b $k < b < i_1$
$\lambda_{i_h} < v \le \lambda_{i_{h-1}}$ $1 \le h \le t+1$ $v \ne \lambda_{i_1} + j-1, \lambda_{i_1}+j$			i_{h-1} -	i_o		
$\lambda_{i_1} + j - 1$		1			$\frac{1}{2}$	0
$\lambda_{i_1} + j$		0			$\frac{1}{2}$	1
$\lambda_{i_o} < v \le \lambda_1$				0		

TABLE (IV)

D $y_o', \Lambda(y_o')$ (E_v) $w, \Lambda(w)$ / v \ Λ	Λ_k^o	Λ_{i_g} $2 \le g \le t$	$\Lambda_{i_{g-1}+1}, \dots, \Lambda_{i_g-1}$ $2 \le g \le t$	$\Lambda_{i_1}^o$	$\Lambda_{i_1}^1$	Λ_a $k < a < i_1$	Λ_k^1	Λ_b $1 \le b < k$
1	0	$(i_t - i_o) + \frac{1}{2}\delta_{gt}$	$i_t - i_o$	$(i_t - i_o) + \frac{1}{2}\delta_{t,1}$	$(i_t - i_o) + 1$	0	$(i_t - i_1) + \frac{1}{2}(1 - \delta_{t1} \cdot \delta_{a_{\lambda,1},1})$	$(i_t - i_1) + 1$
$\lambda_{i_h} + 1$ $3 \le h \le t$	0	$(i_{h-1} - i_o) + \frac{1}{2}(\delta_{gh} + \delta_{g,h-1})$	$(i_{h-1} - i_o) + \delta_{hg}$	$i_{h-1} - i_o$	$(i_{h-1} - i_o) + 1$	0	$(i_{h-1} - i_1) + 1$	
$\lambda_{i_h} + 1 < v \le \lambda_{i_{h-1}}$ $2 \le h \le t+1$ $v \ne \lambda_{i_1}$	0	$i_{h-1} - i_o$			$(i_{h-1} - i_o) + 1$	0	$(i_{h-1} - i_1) + 1$	
$\lambda_{i_2} + 1$ $v \ne 1$	0	$(i_1 - i_o) + \frac{1}{2}\delta_{g,2}$	$(i_1 - i_o) + \delta_{g,2}$	$(i_1 - i_o) + \frac{1}{2}(1 + \delta_{a_{\lambda,1},1})$	$(i_1 - i_o) + 1$	0	$1 - \frac{1}{2}\delta_{a_{\lambda,1},1}$	1
λ_{i_1} $v \ne \lambda_{i_2} + 1$	0	$i_1 - i_o$		$(i_1 - i_o) + \frac{1}{2}$	$(i_1 - i_o) + 1$	0	$\frac{1}{2}$	1
$\lambda_{i_1} + 1$	0				1	0		
$\lambda_{i_1} + 1 < v \le \lambda_{i_o}$	0				1	0		
$\lambda_{i_o} < v \le \lambda_1$	0							

TABLE (IV)

$1 \le j \le \alpha_{\lambda,0}$

D \ A: $y'_j, A(y'_j)$ (E_v) $y_j, A(y_j)$; v	$A^o_{k'}$, A_a $i_1 < a \le r$, $A^o_{i_1}$	$A^1_{i_1}$	A_b $k < b < i_1$	A^1_k	A_c $1 \le c < k$
$\lambda_{i_h} < v \le \lambda_{i_{h-1}}$ $1 < h \le t+1$	$i_{h-1} - i_o$	$i_{h-1} - i_o + 1$	0	$i_{h-1} - i_1 + 1$	
$\lambda_{i_1} < v \le \lambda_{i_o}$ $v \ne \lambda_{i_1} + j$	0	1	0		
$\lambda_{i_1} + j$	1		0		1
$\lambda_{i_o} < v \le \lambda_1$	0				

TABLE (V)

$1 \leq j \leq a_{\lambda,0}$

$D^{y_j,\Lambda(y_j)}_{v,\Lambda(w)}(K_v)$ — Λ / v	Λ_k^0	Λ_{i_g} $2 \leq g \leq t$	$\Lambda_{i_{g-1}+1},\ldots,\Lambda_{i_g-1}$ $2 \leq g \leq t$	$\Lambda_{i_1}^0$	$\Lambda_{i_1}^1$	$\Lambda_{i_0+1},\ldots,\Lambda_{i_1-1}$	Λ_k^1	Λ_a $1 \leq a < k$
1	$(j-1)(i_t-i_o)$	$j(i_t-i_o)+\frac{1}{2}\delta_{gt}$	$j(i_t-i_o)$	$j(i_t-i_o)+\frac{1}{2}\delta_{t1}$	$j(i_t-i_o)+\frac{1}{2}\delta_{t1}\cdot\delta_{a_{\lambda,1},1}$	$j(i_t-i_o)$	$j(i_t-i_o)-\frac{1}{2}$	$j(i_t-i_o)$
$\lambda_{i_h}+1$ $3 \leq h \leq t$	$(j-1)(i_{h-1}-i_o)$	$j(i_{h-1}-i_o)+\frac{1}{2}(\delta_{gh}+\delta_{g,h-1})$	$j(i_{h-1}-i_o)+\delta_{gh}$	$j(i_{h-1}-i_o)$				
$\lambda_{i_h}+1 < v \leq \lambda_{i_{h-1}}$ $2 \leq h \leq t+1$ $v \neq \lambda_{i_1}$	$(j-1)(i_{h-1}-i_o)$	$j(i_{h-1}-i_o)$						
$\lambda_{i_2}+1$ $v \neq 1$	$(j-1)(i_1-i_o)$	$j(i_1-i_o)+\frac{1}{2}\delta_{g,2}$	$j(i_1-i_o)+\delta_{g,2}$	$j(i_1-i_o)+\frac{1}{2}(1+\delta_{a_{\lambda,1},1})$	$j(i_1-i_o)+\frac{1}{2}\delta_{a_{\lambda,1},1}$	$j(i_1-i_o)$		
λ_{i_1} $v \neq \lambda_{i_2}+1$	$(j-1)(i_1-i_o)$	$j(i_1-i_o)$		$j(i_1-i_o)+\frac{1}{2}$	$j(i_1-i_o)+\frac{1}{2}$	$j(i_1-i_o)$		
$\lambda_{i_1}+\ell$ $1 \leq \ell < j-1$	1							
$\lambda_{i_1}+j-1$ $v \neq \lambda_{i_1}$	1							
$\lambda_{i_1}+j$	0				$\frac{1}{2}$	1	$\frac{1}{2}$	0
$\lambda_{i_1}+\ell'$ $j < \ell' \leq a_{\lambda,0}$	0							
$\lambda_{i_o} < v \leq \lambda_{i_1}$	0							

TABLE (VI)

$1 \le j \le a_{\lambda,0}$

$D^{y_j', \Lambda(y_j')}_{v, \Lambda(v)}(E_v)$ — Λ / v	Λ_k^0	Λ_{i_g} $2 \le g \le t$	$\Lambda_{i_{g-1}+1}, \dots, \Lambda_{i_g-1}$ $2 \le g \le t$	$\Lambda^0_{i_1}$	$\Lambda^1_{i_1}$	$\Lambda_{i_o+1}, \dots, \Lambda_{i_1-1}$	Λ^1_k	Λ_a $1 \le a < k$
1	$j(i_t-i_o)$	$(j+1)(i_t-i_o)+\frac{1}{2}\delta_{gt}$	$(j+1)(i_t-i_o)$	$(j+1)(i_t-i_o)+\frac{1}{2}\delta_{t1}$	$(j+1)(i_t-i_o)+1+\frac{1}{2}\delta_{t1}\cdot\delta_{a_{\lambda,1},1}$	$j(i_t-i_o)$	$(j+1)(i_t-i_o)+i_o-i_1+\frac{1}{2}$	$(j+1)(i_t-i_o)+i_o-i_1+1$
$\lambda_{i_h}+1$ $3 \le h \le t$	$j(i_{h-1}-i_o)$	$(j+1)(i_{h-1}-i_o)+\frac{1}{2}(\delta_{gh}+\delta_{g,h-1})$	$(j+1)(i_{h-1}-i_o)+\delta_{gh}$	$(j+1)(i_{h-1}-i_o)$	$(j+1)(i_{h-1}-i_o)+1$	$j(i_{h-1}-i_o)$	$(j+1)(i_{h-1}-i_o)+i_o-i_1+1$	
$\lambda_{i_h}+1<v\le\lambda_{i_{h-1}}$ $2 \le h \le t+1$ $v \ne \lambda_{i_1}$	$j(i_{h-1}-i_o)$	$(j+1)(i_{h-1}-i_o)$			$(j+1)(i_{h-1}-i_o)+1$	$j(i_{h-1}-i_o)$	$(j+1)(i_{h-1}-i_o)+i_o-i_1+1$	
$\lambda_{i_2}+1$ $v \ne 1$	$j(i_1-i_o)$	$(j+1)(i_1-i_o)+\frac{1}{2}\delta_{g,2}$	$(j+1)(i_1-i_o)+\delta_{g,2}$	$(j+1)(i_1-i_o)+\frac{1}{2}(1+\delta_{a_{\lambda,1},1})$	$(j+1)(i_1-i_o)+1+\frac{1}{2}\delta_{a_{\lambda,1},1}$	$j(i_1-i_o)$	$j(i_1-i_o)+1$	
λ_{i_1} $v \ne \lambda_{i_2}+1$	$j(i_1-i_o)$	$(j+1)(i_1-i_o)$		$(j+1)(i_1-i_o)+\frac{1}{2}$	$(j+1)(i_1-i_o)+\frac{3}{2}$	$j(i_1-i_o)$	$j(i_1-i_o)+1$	
$\lambda_{i_1}+\ell$ $1 \le \ell < j-1$	1				2	1		
$\lambda_{i_1}+j-1$ $v \ne \lambda_{i_1}$	1				2	1		
$\lambda_{i_1}+j$	1				$\frac{3}{2}$	1	$\frac{1}{2}$	1
$\lambda_{i_1}+\ell'$ $j < \ell' \le a_{\lambda,0}$	0				1	0		
$\lambda_{i_o} < v \le \lambda_{i_1}$	0							

CHAPTER 10 : RAISING OPERATIONS ON LAYERS

So far, we have considered various operations on the affine Weyl group $\tilde{A}_n$ which are successions of left star operations and which therefore, when applied to an element w, give us another element in the same left cell as w. In the present chapter, we introduce a new type of operation which, when applied to an element w in N_λ, gives us another element of N_λ in the same left cell as w but which is not in general a succession of star operations. This operation is called a raising operation on a layer. It is defined in §10.2 and is proved to give an element in the same left cell in §10.3 and §10.4. By means of a succession of these operations and left star operations, every element of $\tilde{A}_n$ can be transformed into a principal normalized element.

§10.1 REFLECTIVE PAIRS

In this section, we introduce a criterion for elements $y, w \in \tilde{A}_n$ to satisfy $y < w$ and $\ell(w) = \ell(y) + 1$. This criterion will be used subsequently in this chapter.

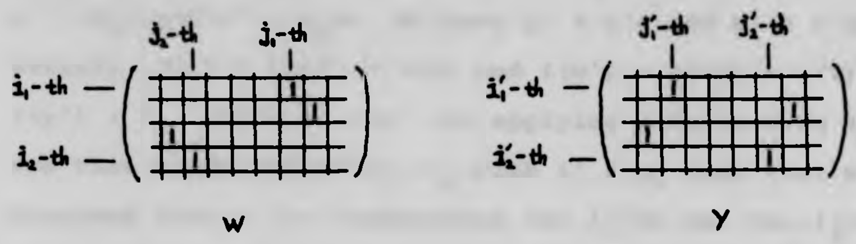
For $x \in \tilde{A}_n$, we call an entry pair $\{e(i_1,j_1), e(i_2,j_2)\}$ of x reflective if the following conditions are satisfied:

(i) either $|i_1-i_2| < n$ or $|j_1-j_2| < n$

(ii) For any entry $e(i,j)$ of x, $(i-i_1)(i-i_2) < 0$ implies $(j-j_1)(j-j_2) > 0$.

In that case, when $(i_1-i_2)(j_1-j_2) > 0$ (resp. $(i_1-i_2)(j_1-j_2) < 0$), $\{e(i_1,j_1), e(i_2,j_2)\}$ is called a positive (resp. negative) reflective pair.

For example, suppose $n > 4$. Then in the following elements $w, y \in A_n$.



$$E_w = \{e_w(i_1,j_1), e_w(i_2,j_2)\}, \quad E_y = \{e_y(i_1',j_1'), e_y(i_2',j_2')\}$$

are reflective pairs. In particular, E_w is negative and E_y positive.

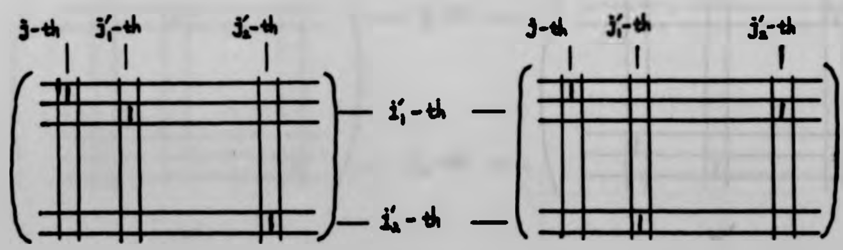
<u>Lemma 10.1.1</u> <u>Let $y, w \in A_n$. Then $y < w$ and $\ell(w) = \ell(y) + 1$ if and only if w is obtained from y by transposing the i_1-th and the i_2-th rows of y for some integers i_1, i_2 with $i_1 \neq i_2$ and $\{e_y(i_1,j_1), e_y(i_2,j_2)\}$ a positive reflective pair.</u>

In the above example, suppose that $(i_1',j_1',i_2',j_2') = (i_1,j_2,i_2,j_1)$ and suppose that w is obtained from y by transposing the i_1-th and the i_2-th rows. Then we can see that $w = s_{i_1+1}s_{i_1}s_{i_1+2}s_{i_1+1}s_{i_1+2}s_{i_1}s_{i_1+1}y$ with $\ell(w) - 4 = \ell(s_{i_1+2}s_{i_1}s_{i_1+1}y) = \ell(y) - 3$. So we have $y < w$ and $\ell(w) = \ell(y) + 1$.

<u>Proof of Lemma 10.1.1</u>

($\Rightarrow$) We can write $y = xz$, $w = xsz$ with $s \in \Delta$ and $x,z \in A_n$ such that $\ell(y) = \ell(x) + \ell(z)$ and $\ell(w) = \ell(x) + \ell(z) + 1$. If $x = 1$, then the result is obvious. Now assume that $x \neq 1$. We write $x = s_v x'$ for some $s_v \in \mathcal{L}(x)$. Then $s_v \in \mathcal{L}(y) \cap \mathcal{L}(w)$. Let $y' = s_v y$ and $w' = s_v w$. We have $y' = x'z$ and $w' = x'sz$ which satisfy $\ell(y') = \ell(x') + \ell(z)$ and $\ell(w') = \ell(x') + \ell(z) + 1 = \ell(y') + 1$. Since $y' < w'$, by applying induction on $\ell(x)$, we see that there exists i'_1, i'_2 with $i'_1 \neq i'_2$ such that w' is obtained from y' by transposing the i'_1-th and the i'_2-th rows of y' with $\{e_{y'}(i'_1,j'_1), e_{y'}(i'_2,j'_2)\}$ a positive reflective pair. Without loss of generality, we may assume $i'_1 < i'_2$. If $\{\bar{v}, \overline{v+1}\} \cap \{\bar{i}'_1, \bar{i}'_2\} = \emptyset$, then w is obtained from y by transposing the i'_1-th and the i'_2-th rows of y with $\{e_y(i'_1,j'_1), e_y(i'_2,j'_2)\}$ a positive reflective pair. If $|\{\bar{v}, \overline{v+1}\} \cap \{\overline{i'_1}, \overline{i'_2}\}| = 1$, then since $s_v \in \mathcal{L}(y) \cap \mathcal{L}(w)$, one of the following four cases must occur.

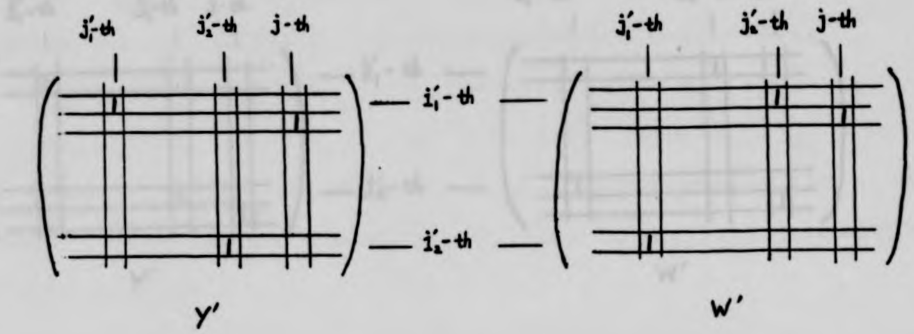
(i)



y' w'

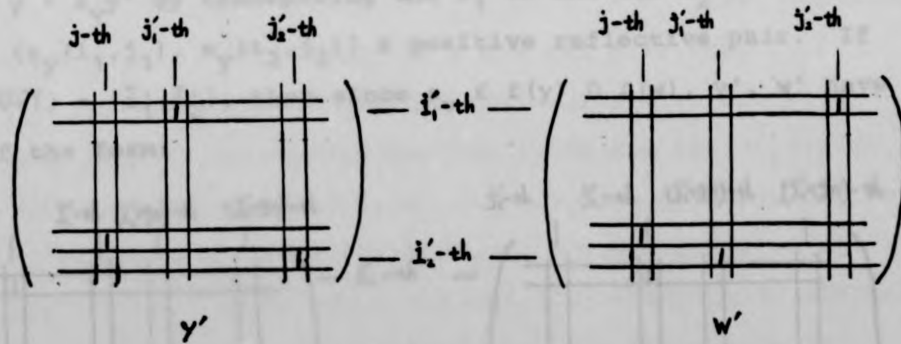
where $\overline{v+1} = \bar{i}'_1$, $\bar{v} \neq \bar{i}'_2$ and $j < j'_1 < j'_2$. Let $(i_1, j_1, i_2, j_2) = (i'_1 - 1, j'_1, i'_2, j'_2)$.

(ii)



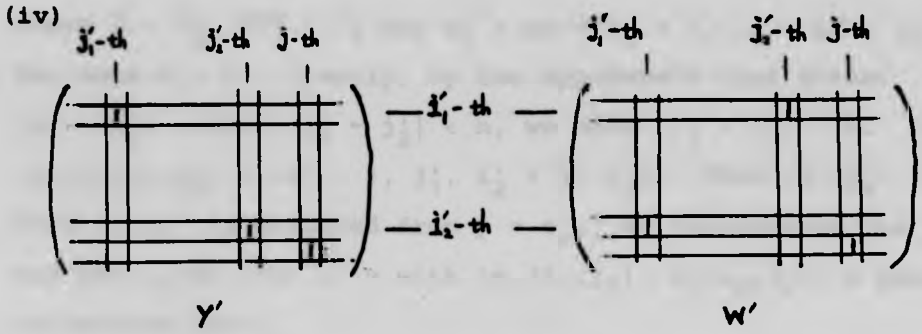
where $\bar{v} = \bar{I}_1'$, $\overline{v+1} \neq \bar{I}_2'$ and $j_1' < j_2' < j$. Let $(i_1, j_1, i_2, j_2) = (i_1' + 1,\ j_1',\ i_2',\ j_2')$.

(iii)

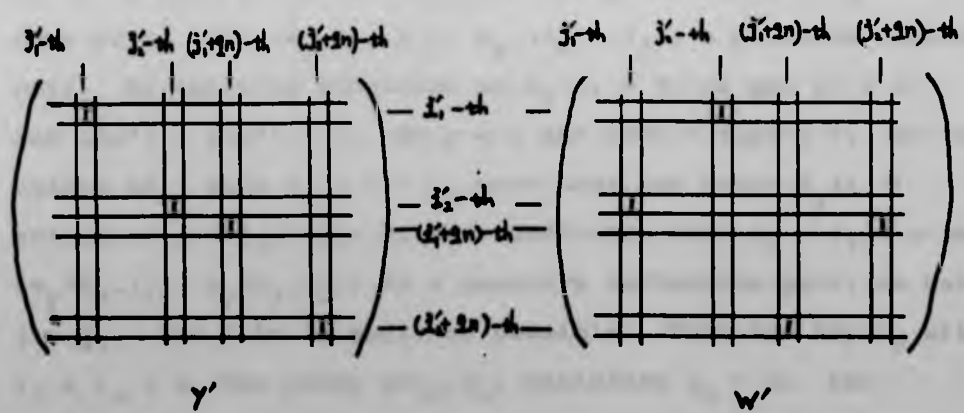


where $\overline{v+1} = \bar{I}_2'$, $\bar{v} \neq \bar{I}_1'$ and $j < j_1' < j_2'$. Let $(i_1,\ j_1,\ i_2,\ j_2) = (i_1',\ j_1',\ i_2'-1,\ j_2')$.

(iv)



where $\bar{v} = \bar{i}_2'$, $\overline{v+1} \neq \bar{i}_1'$ and $j_1' < j_2' < j$. Let $(i_1,j_1,i_2,j_2) = (i_1',j_1', i_2'+1, j_2')$. Clearly, in any of the above cases, we see that $|i_1-i_2| < n$ if and only if $|i_1' - i_2'| < n$ and also that $|j_1-j_2| < n$ if and only if $|j_1'-j_2'| < n$. So $w = s_v w'$ is obtained from $y = s_v y'$ by transposing the i_1-th and the i_2-th rows of y with $\{e_y(i_1,j_1), e_y(i_2,j_2)\}$ a positive reflective pair. If $\{\bar{v}, \overline{v+1}\} = \{\bar{i}_1',\bar{i}_2'\}$, then since $s_v \in \mathcal{L}(y) \cap \mathcal{L}(w)$, y', w' have to be of the form:



where $\bar{v} = \bar{i}'_2$, $\overline{v+1} = \bar{i}'_1$ and $i'_1 + qn = i'_2 + 1$, $j'_1 + qn > j'_2$ for some $q > 0$. Clearly, by the hypothesis that either $|i'_1 - i'_2| < n$ or $|j'_1 - j'_2| < n$, we have $|j'_1 - j'_2| < n$. Let $(i_1, j_1, i_2, j_2) = (i'_1 - 1, j'_1, i'_2 + 1, j'_2)$. Then $|j_1 - j_2| < n$. So $w = s_v w'$ is obtained from $y = s_v y'$ by transposing the i_1-th and the i_2-th rows of y with $\{e_y(i_1,j_1), e_y(i_2,j_2)\}$ a positive reflective pair.

($\Leftarrow$) Since w is also obtained from y by transposing the j_1-th and the j_2-th columns of y, we may assume $0 < i_2 - i_1 < n$ by symmetry. When $i_2 - i_1 = 1$, we have $w = s_{i_1} y$ with $\ell(w) = \ell(y)+1$ and so our result is true. Now assume $i_2 - i_1 > 1$. If, for any i, $i_1 < i < i_2$, the entry $e_y(i,j)$ satisfies $j > j_2$, then $s_{i_2-1} \in \mathcal{L}(y) \cap \mathcal{L}(w)$. Let $y' = s_{i_2-1}y$, $w' = s_{i_2-1}w$. Hence we see that $y' < w'$ if and only if $y < w$ and also that $\ell(w') = \ell(y') + 1$ if and only if $\ell(w) = \ell(y)+1$. But w' is obtained from y' by transposing the i_1-th and the (i_2-1)-th rows of y' with $\{e_{y'}(i_1,j_1), e_{y'}(i_2-1,j_2)\}$ a positive reflective pair. By applying induction on $i_2 - i_1 > 1$, we get $y' < w'$ and $\ell(w') = \ell(y') + 1$. So $y < w$ and $\ell(w) = \ell(y) + 1$. If there exists an i with $i_1 < i < i_2$ such that the entry $e_y(i,j)$ satisfies $j < j_2$, then by the conditions that $i_2 - i_1 < n$ and $\{e_y(i_1,j_1), e_y(i_2,j_2)\}$ is a positive reflective pair, we have $j < j_1$. Let i be as small as possible. Then for any i_o with $i_1 < i_o < i$, the entry $e(i_o,j_o)$ satisfies $j_o > j$. Let $y' = s_{i_1}s_{i_1+1}\cdots s_{i-1}y$ and $w' = s_{i_1}s_{i_1+1}\cdots s_{i-1}w$. We have

$\ell(y') = \ell(y) - (i-i_1)$ and $\ell(w') = \ell(w) - (i-i_1)$. So $y' < w'$ if and only if $y < w$, and also $\ell(w') = \ell(y') + 1$ if and only if $\ell(w) = \ell(y) + 1$. By the same argument as the above, we get $y < w$ and $\ell(w) = \ell(y) + 1$. By induction, our conclusion is proved. □

§10.2 RAISING OPERATIONS ON LAYERS

Let $w \in A_n$ and $i_1,\ldots,i_t$ be integers satisfying $i_1 < \ldots < i_t < i_1 + n$. Let $f(i_h)$ be the i_h-th row of w for $1 < h < t$.

Procedure (i): Let $B_1,\ldots,B_t$ be the blocks of w such that for every h, $1 < h < t$, B_h consists of rows of w from the $(i_{h-1}+1)$-th to the i_h-th and B_1 from the (i_t+1-n)-th to the i_1-th.

Let $w' \in A_n$ be obtained from w by permuting $f(i_h)$ from the bottom to the top in B_h for all h, $1 < h < t$.

Procedure (ii): Let $B_1',\ldots,B_t'$ be the blocks of w such that for every h, $1 < h < t$, B_h' consists of rows of w from the i_h-th to the $(i_{h+1}-1)$-th and B_t' from the i_t-th to the (i_1-1+n)-th.

Let $w'' \in A_n$ be obtained from w by permuting $f(i_h)$ from the top to the bottom in B_h' for all h, $1 < h < t$.

Let F be the set of row classes of w consisting of $\{f(i_1),\ldots,f(i_t)\}$ and their congruent rows. Then procedure (i) is called "raising F by one term" and (ii) called "lowering F by one term". Let $\bar{e}(w)$ be the set of entry classes of w

consisting of all entries lying in F. Then we can also say that procedure (i) raises $\bar{e}(w)$ by one term and (ii) lowers $\bar{e}(w)$ by one term. So w' is obtained from w by raising $\bar{e}(w)$ by one term and w" from w by lowering $\bar{e}(w)$ by one term. Let $E \subseteq \underline{n}$ be such that $E = \{\bar{h} \mid \text{there exists some } e_w(i,h) \in \bar{e}(w)\}$, then we write $w' = L_E(w)$, $w" = L^E(w)$ and write $\bar{e}(w) = \bar{e}((w),E)$. Clearly, $L_E.L^E(w) = L^E.L_E(w) = w$.

Assume that $w \in \mathbb{N}_\lambda$ has a standard MDC form $(A_r,\ldots,A_1)$ at $i \in \mathbb{Z}$. Let $E_u \subseteq \underline{n}$ for $1 < u < \lambda_1$ such that $E_u = \{\overline{j_t^u(w)} \mid 1 < u < \mu_u\}$, where $\lambda = \{\lambda_1 > \ldots > \lambda_r\} \in \Lambda_n$ and $\mu = \{\mu_1 > \ldots > \mu_m\}$ is the dual partition of λ.

<u>Definition 10.2.1</u> For any u, $1 < u < \lambda_1$, the u-th layer $\bar{e}((w),E_u)$ of w is movable if $L_{E_u}(w) \in \mathbb{N}_\lambda$ and immovable otherwise.

Clearly, the layer $\bar{e}((w),E_1)$ is always movable. $w \in \bar{\mathbb{N}}_\lambda$ if and only if the layers $\bar{e}((w),E_u)$, $1 < u < \lambda_1$, are all immovable.

When $w' = L_{E_u}(w) \in \mathbb{N}_\lambda$ (resp. $w" = L^{E_u}(w) \in \mathbb{N}_\lambda$) for some u, $1 < u < \lambda_1$, we let $(A_r,\ldots,A_1)$ be the standard MDC form of w' at i+1 (resp. of w" at i-1).

Clearly, for any v, $1 < v < \lambda_1$, the layer $\bar{e}((w'),E_v)$ is an increasing chain of w' in that case. So we can work out the following $D_w^{w'}$-table.

Table (VII)

$D^{w',A(w')}_{w,A(w)}(E_u)$ — A / u	A_t , $A_{t,h}$, $A'_{t,h}$ $1 < t < r$, $1 < h < \lambda_t$
$1 < u < \lambda_1$ $u \neq v$	o
v	1

where $A_{t,h}$ is the block consisting of the first h rows of A_t, $A'_{t,h}$ is the block consisting of the last λ_t-h rows of A_t.

Lemma 10.2.2 Let $w \in \mathbb{N}_\lambda$ and $E_u \subseteq \underline{n}$ be defined as above. Then there exists a sequence of elements $w_o = w, w_1, \ldots, w_\ell$ in $\mathbb{N}_\lambda$ such that for each i, $1 < i < \ell$, $w_i = L_{E_{u_i}}(w_{i-1})$ for some u_i, $1 < u_i < \lambda_1$, and $w_\ell \in \bar{\mathbb{N}}_\lambda$.

Proof If the layers $\bar{e}((w),E_u)$ are immovable for all u with $1 < u < \lambda_1$, then $w \in \bar{\mathbb{N}}_\lambda$ and the result is trivial. Now assume that w are not in such a case. Then there exists a smallest number k with $1 < k < \lambda_1$ such that the layer $\bar{e}((w),E_k)$ is movable. By definition, we have $w_1 = L_{E_k}(w) \in \mathbb{N}_\lambda$. w_1 has a standard MDC form $(A_r,\ldots,A_1)$ at i+1 with

$$\begin{cases} j_t^u(w_1) = j_t^u(w) \text{ for any } u,t \text{ with } 1 \leq t \leq r \text{ and} \\ \qquad\qquad 1 \leq u \leq \min\{k-1,\lambda_t\} \\ j_t^{k-1}(w) > j_t^k(w_1) > j_t^k(w) \text{ for any } t \text{ with } 1 \leq t \leq r \text{ and } k \leq \lambda_t. \end{cases}$$

Clearly, the layers $\bar{e}((w_1),E_h)$, $1 < h < k$, are immovable. If $\bar{e}((w_1),E_k)$ is still movable, then the same procedure could be carried on. We then get $w_o = w.\ w_1,\dots,$ in N_λ such that w_h has a standard MDC form $(A_r,\dots,A_1)$ at $i+h$ for $h = 0,1,2,\dots,$ satisfying

$$\begin{cases} j_t^u(w) = j_t^u(w_1) = j_t^u(w_2) = \dots \text{ for any } t,u \text{ with } 1 \leq t \leq r \\ \qquad\qquad\qquad\qquad \text{and } 1 \leq u \leq \min\{k-1,\lambda_t\} \\ j_t^k(w) < j_t^k(w_1) < j_t^k(w_2) < \dots \\ j_t^k(w_h) < j_t^{k-1}(w) \quad \text{for any } h > 0 \text{ and } t \text{ with } 1 \leq t \leq r \\ \qquad\qquad\qquad \text{and } k \leq \lambda_t. \end{cases}$$

So after a finite number of steps, such a procedure has to stop. i.e. there exists $h > 0$ such that $w_h \in N_\lambda$ has a standard MDC form $(A_r,\dots,A_1)$ at i' and the layers $\bar{e}((w_h),E_u)$, $1 \leq u \leq k$, are all immovable. By applying induction on $\lambda_1-k \geq 0$, our result follows. □

In the following two sections, we shall show:

<u>Proposition 10.2.3</u> <u>Let $w \in N_\lambda$ and $E_u \subseteq \underline{n}$, $1 \leq u \leq \lambda_1$, be defined as in Lemma 10.2.2. If $w' = L_{E_u}(w)$ lies in N_λ, then we have $w' \underset{L}{\sim} w$.</u>

By Lemma 10.2,2, if Proposition 10.2.3 is true, then it is immediate that:

<u>Proposition 10.2.4</u> <u>For any $w \in N_\lambda$, there exists $y \in \bar{N}_\lambda$ such that $y \underset{L}{\sim} w$.</u> □

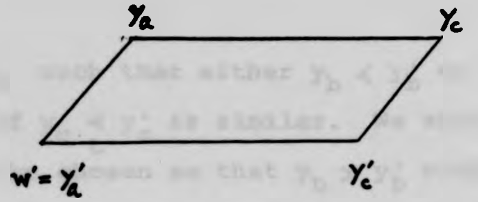
Furthermore, by combining Proposition 10.2.4 with Propositions 6.3.7 and 7.4, we see that:

<u>Proposition 10.2.5</u> <u>For any $w \in \sigma^{-1}(\lambda)$, there exists $y \in \bar{N}_\lambda$ such that $y \underset{L}{\sim} w$.</u> □

§10.3 <u>PROOF OF PROPOSITION 10.2.3 WHEN $1 < u < \lambda_r$.</u>

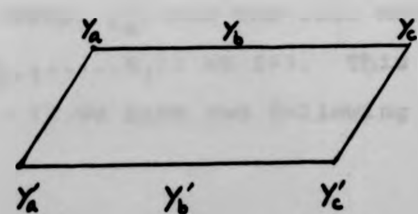
In this section, we wish to show that if $w, w' \in N_\lambda$ and w' is obtained from w by raising the u-th layer where $1 < u < \lambda_r$, then w' lies in the same left cell as w. The idea for proving this will be as follows.

We consider the sequences $\xi(w,r)$, $\xi(w',r)$ beginning with w, w', respectively. We pick a certain element y_a in the sequence $\xi(w,r)$ and define $y_a' = w'$. We define y_c to be the last term in the sequence $\xi(w,r)$ and define y_c' to be the element in the sequence $\xi(w',r)$ obtained by applying the same sequence of left star operations to w' as are needed to obtain y_c from y_a. We then consider the cycle of elements shown in the figure



We show that $y'_a \underset{L}{<} y_a$ and $y_c \underset{L}{<} y'_c$. It follows that all the elements in this cycle lie in the same left cell. Since y_a lies in the same left cell as w and $w' = y'_a$, we see that w, w' are in the same left cell.

In order to show $y'_a \underset{L}{<} y_a$ and $y_c \underset{L}{<} y'_c$, we argue as follows. We find a certain element y_b in the sequence $\xi(w,r)$ which lies between y_a and y_c, and define y'_b to be the element in $\xi(w',r)$ obtained by applying the same succession of left star operations to y'_a as are needed to obtain y_b from y_a in $\xi(w,r)$. We thus have a figure



To show $y'_a \underset{L}{<} y_a$, it is enough to show the two statements. (i) $\mathcal{L}(y'_a) \not\subseteq \mathcal{L}(y_a)$. (ii) Either $y'_a \prec y_a$ or $y_a \prec y'_a$. Statement (i) will follow readily by observing the full MDC form of y'_a and y_a. In order to prove statement (ii) it will be sufficient to find y_b, y_b' such that either $y_b \prec y_b'$ or $y_b' \prec y_b$ by Theorem B. The proof of $y_c \underset{L}{<} y_c'$ is similar. We shall show in fact that y_b, y_b' can be chosen so that $y_b > y_b'$ with $\ell(y_b) = \ell(y_b') + 1$ or $y_b' > y_b$ with $\ell(y_b') = \ell(y_b) + 1$. This will certainly imply $y_b \succ y_b'$ or $y_b' \succ y_b$.

Now let us start our proof. Assume that $w \in N_\lambda$ has a standard MDC form $(A_r,\ldots,A_1)$ at $i \in \mathbb{Z}$. Let A_r^o (resp. A_r^1) be the block consisting of the first row (resp. the last λ_r-1 rows) of A_r.

If $\lambda_r = 1$, then $w' = \rho^{A_r}_{A_1,\ldots,A_{r-1}}(w)$ and the result is true. So we may assume $\lambda_r > 1$.

Let $y_a = \rho^{A_r^o}_{A_1,\ldots,A_{r-1},A_r^1}(w)$, $y_a' = w'$. Then y_a lies in the sequence $\xi(w,r)$ beginning with w and the $D_w^{y_a}$-table is obtained from Table (I) by substituting y_a, w for x_2, x_1. Clearly, y_a (resp. y_a') has the full MDC form $(A_r^o,A_r^1,A_{r-1},\ldots,A_1)$ (resp. $(A_r,A_{r-1},\ldots,A_1)$) at $i+1$. This implies $\mathcal{L}(y_a') \not\subseteq \mathcal{L}(y_a)$.

When $u = 1$, we have the following D-tables.

$D_{w,A(w)}^{y_a,A(y_a)}(E_v)$ v \ A	$A_1,\ldots,A_{r-1}$	A_r^1	A_r^o
$3 < v < \lambda_1$	0	0	0
1	1	$\frac{1}{2}$	$\frac{1}{2}$
2	0	$\frac{1}{2}$	$\frac{1}{2}$

$D_w^{y_a}$-Table

$D_{w,A(w)}^{y_a',A(y_a')}(E_v)$ v \ A	$A_1,\ldots,A_{r-1}$	A_r^1	A_r^o
$3 < v < \lambda_1$	0	0	0
1	1	1	1
2	0	0	0

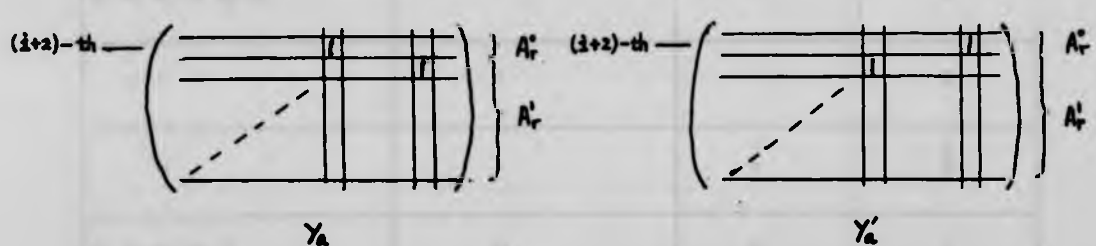
$D_w^{y_a'}$-Table

Then by Lemma 9.8.2, we get

$D_{y_a,A(y_a)}^{y_a',A(y_a')}(E_v)$ v \ A	$A_1,\ldots,A_{r-1}$	A_r^1	A_r^o
$3 < v < \lambda_1$	0	0	0
1	0	$\frac{1}{2}$	$\frac{1}{2}$
2	0	$-\frac{1}{2}$	$-\frac{1}{2}$

$D_{y_a}^{y_a'}$-Table

By tables $D_w^{y_a}$, $D_w^{y'_a}$, we see that $e((y_a), j^1_{A^o_r}(y_a)) \in \bar{e}((y_a), E_2)$ and $e((y'_a), j^1_{A^o_r}(y'_a)) \in \bar{e}((y'_a), E_1)$. Then by table $D_{y_a}^{y'_a}$ and the full DC forms of y_a, y'_a at $i+1$, we see that y'_a is obtained from y_a by transposing the $(i+2)$-th and the α-th rows, where the α-th row of y_a contains an entry belonging to $\bar{e}((y_a), E_1)$ and also to $A^1_r(y_a)$. So by Lemma 9.7.3, y_a, y'_a must have the following forms



Thus by Lemma 10.1.1, we have $y'_a > y_a$ with $\ell(y'_a) = \ell(y_a) + 1$ and hence $y'_a \succ y_a$. So $y'_a \underset{L}{<} y_a$.

When $u > 1$, let

$$\begin{cases} y_b = \wp^{A^o_r}_{A_1,\dots,A_{r-1}} \left(\wp^{A^o_r}_{A_1,\dots,A_{r-1},A^1_r}\right)^{u-1}(w) = \wp^{A^o_r}_{A_1,\dots,A_{r-1}} \left(\wp^{A^o_r}_{A_1,\dots,A_{r-1},A^1_r}\right)^{u-2}(y_a) \\ y'_b = \wp^{A^o_r}_{A_1,\dots,A_{r-1}} \left(\wp^{A^o_r}_{A_1,\dots,A_{r-1},A^1_r}\right)^{u-2}(y'_a). \end{cases}$$

Then y_b (resp. y'_b) has the full DC form $(A^1_r, A^o_r, A_{r-1}, \dots, A_1)$ at $i+u$ which is an element of the sequence $\xi(w,r)$ (resp. $\xi(y'_a, r)$) beginning with w (resp. y'_a). y'_b is obtained from y'_a by applying the same succession of left star operations as are needed to obtain y_b from y_a in $\xi(w,r)$. From Tables (II),

(III) and Lemma 9.8.2, we get

$D_w^{y_b}$-Table

$D_{w,A(w)}^{y_b,A(y_b)}(E_v)$ A / v	$A_1,\dots,A_{r-1}$	A_r^1	A_r^o
1	1	$\frac{1}{2}$	$\frac{3}{2}$
$2 < v < u-2$	1	1	2
$u-1$	1	1	2
u	1	$\frac{1}{2}$	$\frac{3}{2}$
$u < v < \lambda_r$	0	0	1
$\lambda_r < v < \lambda_1$	0	0	0

$D_{w}^{y_b'}$-Table

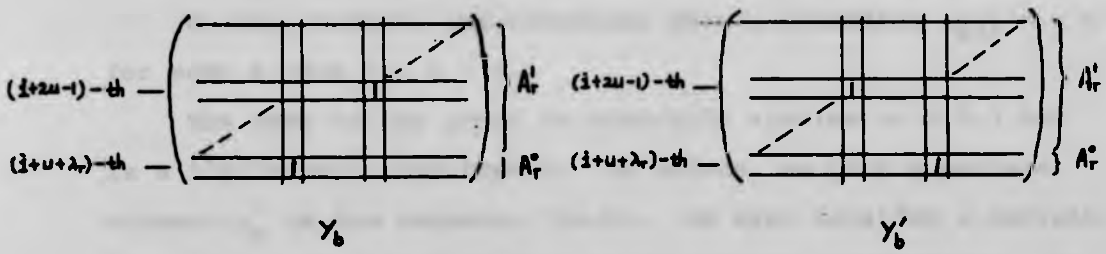
$D_{w,A(w)}^{y_b',A(y_b')}(E_v)$ $v \backslash A$	$A_1,\ldots,A_{r-1}$	A_r^1	A_r^o
1	1	$\frac{1}{2}$	$\frac{3}{2}$
$2 < v < u-2$	1	1	2
$u-1$	1	$\frac{1}{2}$	$\frac{3}{2}$
u	1	1	2
$u < v < \lambda_r$	o	o	1
$\lambda_r < v < \lambda_1$	o	o	o

Hence by Lemma 9.8.2, we have

$D_{y_b,A(y_b)}^{y_b',A(y_b')}(E_v)$ $v \backslash A$	$A_1,\ldots,A_{r-1}$	A_r^1	A_r^o
$1 < v < \lambda_1$ $v \neq u-1,u$	o	o	o
$u-1$	o	$-\frac{1}{2}$	$-\frac{1}{2}$
u	o	$\frac{1}{2}$	$\frac{1}{2}$

$D_{y_b}^{y_b'}$-Table

By tables $D_w^{y_b}$, $D_w^{y'_b}$, we see that $e((y_b), j^1_{A_r^o}(y_b)) \in \bar{e}((y_b), E_u)$ and $e((y'_b), j^1_{A_r^o}(y'_b)) \in \bar{e}((y'_b), E_{u-1})$. Then by Table $D_{y_b}^{y'_b}$ and the full DC forms of y_b, y'_b at $i+u$, we see that y'_b is obtained from y_b by transposing the α-th and the $(i+u+\lambda_r)$-th rows, where the α-th row of y_b contains an entry belonging to $\bar{e}((y_b), E_{u-1})$ and also to $A_r^1(y_b)$. So by Lemma 9.7.3, y_b, y'_b must have the following forms



Then by Lemma 10.1.1, we have $y_b > y'_b$ and $\ell(y_b) = \ell(y'_b) + 1$ and hence $y_b \succ y'_b$. Therefore, either $y'_a \succ y_a$ or $y'_a \prec y_a$ holds by Theorem B. This implies $y'_a \underset{L}{<} y_a$.

On the other hand, let

$$\begin{cases} y_c = \wp^{A_r^o}_{A_1,\dots,A_{r-1}} \left(\wp^{A_r^o}_{A_1,\dots,A_{r-1},A_r^1}\right)^{\lambda_r-1}(w) = \wp^{A_r^o}_{A_1,\dots,A_{r-1}} \left(\wp^{A_r^o}_{A_1,\dots,A_{r-1},A_r^1}\right)^{\lambda_r-2}(y_a) \\ y'_c = \wp^{A_r^o}_{A_1,\dots,A_{r-1}} \left(\wp^{A_r^o}_{A_1,\dots,A_{r-1},A_r^1}\right)^{\lambda_r-2}(y'_a). \end{cases}$$

Then y_c is the last term of the sequence $\xi(w,r)$ but y'_c is not the last term of $\xi(y'_a,r)$. So y_c has the full MDC form $([A_r^1, A_r^o], A_{r-1}, \dots, A_1)$ at $i + \lambda_r$ and y'_c has the full MDC form

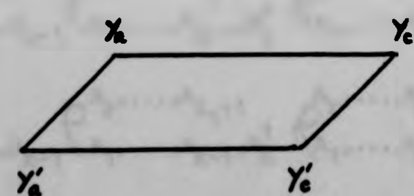
$(A_r^1, A_r^0, A_{r-1}, \ldots, A_1)$ at $i + \lambda_r$. This implies $\mathcal{L}(y_c) \not\subseteq \mathcal{L}(y_c')$. Now y_c' is obtained from y_a' by the same left star operations as are needed to obtain y_c from y_a in $\xi(w,r)$. According to Theorem B, either $y_c \succ y_c'$ or $y_c \prec y_c'$ holds and so $y_c \underset{L}{\leq} y_c'$.

Therefore, we have $y_a \underset{P_L}{\sim} y_c \underset{L}{\leq} y_c' \underset{P_L}{\sim} y_a' \underset{L}{\leq} y_a$ and this implies $y_a \underset{L}{\sim} y_a'$. So $w \underset{P_L}{\sim} y_a$ and $w' = y_a'$ imply $w \underset{L}{\sim} w'$.

§10.4 PROOF OF PROPOSITION 10.2.3 WHEN $\lambda_{k+1} < u < \lambda_k$ AND $1 < k < r$.

We now consider the situation when u satisfies $\lambda_{k+1} < u < \lambda_k$ for some k with $1 < k < r$.

The idea of the proof is generally similar to §10.3 but is a little more complicated. As before, we pick a certain element y_a in the sequence $\xi(w,k)$. We also consider a certain element $y_a' \in \mathfrak{N}_\lambda$ obtained from w' by certain ρ-operations. We compare the sequences $\xi(w,k)$ and $\xi(y_a',k)$. Let y_c be the last term of the sequence $\xi(w,k)$ and y_c' be the term in $\xi(y_a',k)$ obtained by applying the same sequence of left star operations to y_a' as are needed to obtain y_c from y_a in $\xi(w,k)$. We then have a cycle of elements as in the figure



We show that all the elements in this cycle lie in the same left cell as before by finding corresponding elements y_b, y'_b in the two sequences satisfying $y_b > y'_b$ and $\ell(y_b) = \ell(y'_b) + 1$. Since $y_a \underset{L}{\sim} w$ and $y'_a \underset{L}{\sim} w'$, we obtain $w \underset{L}{\sim} w'$ as required.

Now we start to prove our result.

Assume that w has a standard MDC form $(A_r,\dots,A_1)$ at $i \in \mathbb{Z}$ and let blocks A^0_k, A^1_k, $A^0_{i_1}$ and $A^1_{i_1}$ be defined as in §9.6.

Let $y_a = \wp^{A_r,\dots,A_{k+1},A^0_k}_{A_1,\dots,A_{k-1},A^1_k}(w)$, $y'_a = \wp^{A_r,\dots,A_{k+1}}_{A_1,\dots,A_k}(w')$. Then y'_a lies in $\aleph_\lambda$ and y_a is in the sequence $\xi(w,k)$ beginning with w. Both y_a and y'_a have the full DC form $(A_r,\dots,A_{k+1},A^0_k,A^1_k,A_{k-1},\dots,A_1)$ at i', $i' = i + \sum_{j=k+1}^{r} \lambda_j + 1$.

Since, by Lemma 9.6.1, $[A^0_k(y'_a), A^1_k(y'_a)]$ is a DC block but $[A^0_k(y_a),A^1_k(y_a)]$ is not, this implies that $\mathcal{L}(y'_a) \not\subseteq \mathcal{L}(y_a)$.

For $u' = u-\lambda_{i_1}$, let

$$\left\{\begin{aligned}
y_b &= \wp^{A^0_{i_1}}_{A^1_k}\wp^{A_{i_1}}_{A_1,\dots,A_{k-1}}\wp^{A^0_k,A_r,\dots,A_{i_1+1}}_{A_1,\dots,A_{k-1},A^1_k}\left(\wp^{A^0_k,A_r,\dots,A_{k+1}}_{A_1,\dots,A_{k-1},A^1_k}\right)^{u'-1}\wp^{A_r,\dots,A_{k+1}}_{A_1,\dots,A_{k-1},A^1_k}(w)\\
&= \wp^{A^0_{i_1}}_{A^1_k}\wp^{A_{i_1}}_{A_1,\dots,A_{k-1}}\wp^{A_r,\dots,A_{i_1+1}}_{A_1,\dots,A_{k-1},A^1_k}\left(\wp^{A_r,\dots,A_{k+1},A^0_k}_{A_1,\dots,A_{k-1},A^1_k}\right)^{u'-1}(y_a)\\
y'_b &= \wp^{A^0_{i_1}}_{A^1_k}\wp^{A_{i_1}}_{A_1,\dots,A_{k-1}}\wp^{A_r,\dots,A_{i_1+1}}_{A_1,\dots,A_{k-1},A^1_k}\left(\wp^{A_r,\dots,A_{k+1},A^0_k}_{A_1,\dots,A_{k-1},A^1_k}\right)^{u'-1}(y'_a)
\end{aligned}\right.$$

Then y_b (resp. y'_b) is an element of the sequence $\xi(w,k)$ (resp. $\xi(y'_a,k)$) beginning with w (resp. y'_a). y'_b is obtained by applying the same sequence of left star operations to y_a' as are needed to obtain y_b from y_a in $\xi(w,k)$. The tables for functions $D_w^{y_b}$ and $D_w^{y_b'}$ can be worked out by Example 9.8.5(i), Tables (IV), (VI), (VII) and Lemma 9.8.2 as follows.

$D_v^{Y_b}$ - Table

$D^{Y_b \in A(Y_b)}_{v,\,A(v)}(E_v)$ A / v	A^0_k	A_{l_g} $2 \le g \le t$	$A_{l_{g-1}+1}, \ldots, A_{l_g-1}$ $2 \le g \le t$	$A^0_{l_1}$	$A^1_{l_1}$	$A_{l_0+1}, \ldots, A_{l_1-}$	A^1_k	A_a $1 \le a < k$
1	$u'(l_t-l_o)$	$(u'+1)(l_t-l_o)+\frac{1}{2}\delta_{gt}$	$(u'+1)(l_t-l_o)$	$(u'+1)(l_t-l_o)+\frac{1}{2}\delta_{1t}$	$(u'+1)(l_t-l_o)+1+\frac{1}{2}\delta_{1t}\cdot\delta_{a_{\lambda,1},1}$	$u'(l_t-l_o)$	$(u'+1)(l_t-l_o)+l_o-l_1+\frac{1}{2}$	$(u'+1)(l_t-l_o)+l_o-l_1+1$
$\lambda_{l_h}+1$ $3 \le h \le t$	$u'(l_{h-1}-l_o)$	$(u'+1)(l_{h-1}-l_o)+\frac{1}{2}(\delta_{hg}+\delta_{h-1,g})$	$(u'+1)(l_{h-1}-l_o)+\delta_{hg}$	$(u'+1)(l_{h-1}-l_o)$	$(u'+1)(l_{h-1}-l_o)+1$	$u'(l_{h-1}-l_o)$	$(u'+1)(l_{h-1}-l_o)+l_o-l_1+1$	
$\lambda_{l_h}+1<v \le \lambda_{l_{h-1}}$ $2 \le h \le t+1$ $v \ne \lambda_{l_1}$	$u'(l_{h-1}-l_o)$	$(u'+1)(l_{h-1}-l_o)$			$(u'+1)(l_{h-1}-l_o)+1$	$u'(l_{h-1}-l_o)$	$(u'+1)(l_{h-1}-l_o)+l_o-l_1+1$	
$\lambda_{l_2}+1$ $v \ne 1$	$u'(l_1-l_o)$	$(u'+1)(l_1-l_o)+\frac{1}{2}\delta_{2,g}$	$(u'+1)(l_1-l_o)+\delta_{2,g}$	$(u'+1)(l_1-l_o)+\frac{1}{2}(1+\delta_{a_{\lambda,1},1})$	$(u'+1)(l_1-l_o)+1+\frac{1}{2}\delta_{a_{\lambda,1},1}$	$u'(l_1-l_o)$	$u'(l_1-l_o)+1$	
λ_{l_1} $v \ne \lambda_{l_2}+1$	$u'(l_1-l_o)$	$(u'+1)(l_1-l_o)$		$(u'+1)(l_1-l_o)+\frac{1}{2}$	$(u'+1)(l_1-l_o)+\frac{3}{2}$	$u'(l_1-l_o)$	$u'(l_1-l_o)+1$	
$\lambda_{l_1}+l$ $1 \le l < u'-1$	1				2	1		
$\lambda_{l_1}+u'-1$ $v \ne \lambda_{l_1}$	1				2	1		
$\lambda_{l_1}+u'$	1				$\frac{3}{2}$	1	$\frac{1}{2}$	1
$\lambda_{l_1}+l'$ $u'<l' \le a_{\lambda,0}$	0				1	0		
$\lambda_{l_0}<v \le \lambda_1$	0							

$D_w^{y_b'}$ - table (when u' = 1)

$D_{w,\Lambda(w)}^{y_b',\Lambda(y_b')}$ (E_v) ; Λ / v	Λ_k^o	Λ_{i_g} $2 \le g \le t$	$\Lambda_{i_{g-1}+1},\dots,\Lambda_{i_g-1}$ $2 \le g \le t$	$\Lambda_{i_1}^o$	$\Lambda_{i_1}^1$	Λ_a $k<a<i_1$	Λ_k^1	Λ_b $1 \le b < k$
1	i_t-i_o	$2(i_t-i_o)+\frac{1}{2}\delta_{gt}$	$2(i_t-i_o)$	$2(i_t-i_o)+\frac{1}{2}\delta_{1,t}$	$2(i_t-i_o)+1$	i_t-i_o	$2(i_t-i_o)+i_o-i_1+\frac{1}{2}(1-\delta_{1t}\cdot\delta_{a_{\lambda,1},1})$	$2(i_t-i_o)+i_o-i_1+1$
$\lambda_{i_h}+1$ $3 \le h \le t$	$i_{h-1}-i_o$	$2(i_{h-1}-i_o)+\frac{1}{2}(\delta_{gh}+\delta_{g,h-1})$	$2(i_{h-1}-i_o)+\delta_{hg}$	$2(i_{h-1}-i_o)$	$2(i_{h-1}-i_o)+1$	$i_{h-1}-i_o$	$2(i_{h-1}-i_o)+i_o-i_1+1$	
$\lambda_{i_h}+1<v\le\lambda_{i_{h-1}}$ $2 \le h \le t+1$ $v \ne \lambda_{i_1}$	$i_{h-1}-i_o$	$2(i_{h-1}-i_o)$			$2(i_{h-1}-i_o)+1$	$i_{h-1}-i_o$	$2(i_{h-1}-i_o)+i_o-i_1+1$	
$\lambda_{i_2}+1$ $v \ne 1$	i_1-i_o	$2(i_1-i_o)+\frac{1}{2}\delta_{g,2}$	$2(i_1-i_o)+\delta_{g,2}$	$2(i_1-i_o)+\frac{1}{2}(1+\delta_{a_{\lambda,1},1})$	$2(i_1-i_o)+1$	i_1-i_o	$(i_1-i_o)+(1-\frac{1}{2}\delta_{a_{\lambda,1},1})$	$(i_1-i_o)+1$
λ_{i_1} $v \ne \lambda_{i_2}+1$	i_1-i_o	$2(i_1-i_o)$		$2(i_1-i_o)+\frac{1}{2}$	$2(i_1-i_o)+1$	i_1-i_o	$(i_1-i_o)+\frac{1}{2}$	$(i_1-i_o)+1$
$\lambda_{i_1}+1$	1				2	1		
$\lambda_{i_1}+1<v\le\lambda_{i_o}$	0				1	0		
$\lambda_{i_o}<v\le\lambda_1$	0							

$D_w^{y'_b}$ - TABLE (when $a_{\lambda,0} \geq u' > 1$)

$D_{v,\Lambda(w)}^{y'_b,\Lambda(y'_b)}(\bar{E}_v)$ — Λ / v	Λ_k^0	Λ_{i_g} $2 \leq g \leq t$	$\Lambda_{i_{g-1}+1},\ldots,\Lambda_{i_g}-1$ $2 \leq g \leq t$	$\Lambda_{i_1}^0$	$\Lambda_{i_1}^1$	$\Lambda_{i_0+1},\ldots,\Lambda_{i_1}-1$	Λ_k^1	Λ_a $1 \leq a < k$
1	$u'(i_t-i_o)$	$(u'+1)(i_t-i_o)+\frac{1}{2}\delta_{gt}$	$(u'+1)(i_t-i_o)$	$(u'+1)(i_t-i_o)+\frac{1}{2}\delta_{1t}$	$(u'+1)(i_t-i_o)+1+\frac{1}{2}\delta_{1t}\cdot\delta_{a_{\lambda,1},1}$	$u'(i_t-i_o)$	$(u'+1)(i_t-i_o)+i_o-i_1+\frac{1}{2}$	$(u'+1)(i_t-i_o)+i_o-i_1+1$
$\lambda_{i_h}+1$ $3 \leq h \leq t$	$u'(i_{h-1}-i_o)$	$(u'+1)(i_{h-1}-i_o)+\frac{1}{2}(\delta_{gh}+\delta_{g,h-1})$	$(u'+1)(i_{h-1}-i_o)+\delta_{gh}$	$(u'+1)(i_{h-1}-i_o)$	$(u'+1)(i_{h-1}-i_o)+1$	$u'(i_{h-1}-i_o)$	$(u'+1)(i_{h-1}-i_o)+i_o-i_1+1$	
$\lambda_{i_h}+1<v\leq\lambda_{i_{h-1}}$ $2 \leq h \leq t+1$ $v \neq \lambda_{i_1}$	$u'(i_{h-1}-i_o)$	$(u'+1)(i_{h-1}-i_o)$			$(u'+1)(i_{h-1}-i_o)+1$	$u'(i_{h-1}-i_o)$	$(u'+1)(i_{h-1}-i_o)+i_o-i_1+1$	
$\lambda_{i_2}+1$ $v \neq 1$	$u'(i_1-i_o)$	$(u'+1)(i_1-i_o)+\frac{1}{2}\delta_{g,2}$	$(u'+1)(i_1-i_o)+\delta_{g,2}$	$(u'+1)(i_1-i_o)+\frac{1}{2}(1+\delta_{a_{\lambda,1},1})$	$(u'+1)(i_1-i_o)+1+\frac{1}{2}\delta_{a_{\lambda,1},1}$	$u'(i_1-i_o)$	$u'(i_1-i_o)+1$	
λ_{i_1} $v \neq \lambda_{i_2}+1$	$u'(i_1-i_o)$	$(u'+1)(i_1-i_o)$		$(u'+1)(i_1-i_o)+\frac{1}{2}$	$(u'+1)(i_1-i_o)+\frac{3}{2}$	$u'(i_1-i_o)$	$u'(i_1-i_o)+1$	
$\lambda_{i_1}+\ell$ $1 \leq \ell < u'-1$	1				2	1		
$\lambda_{i_1}+u'-1$ $v \neq \lambda_{i_1}$	1				$\frac{3}{2}$	1	$\frac{1}{2}$	1
$\lambda_{i_1}+u'$	1				2	1		
$\lambda_{i_1}+\ell'$ $u'<\ell'\leq a_{\lambda,0}$	0				1	0		
$\lambda_{i_o}<v\leq\lambda_1$	0							

Then also by Lemma 9.8.2, we get the following table.

$D^{y_b', A(y_b')}_{y_b, A(y_b)}(E_v)$; v \ A	A_h, $A^0_{i_1}$, A^0_k $1 < h < r$, $h \neq k, i_1$	$A^1_{i_1}$	A^1_k
$1 < v < \lambda_1$ $v \neq u-1, u$	0	0	0
$u - 1$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
u	0	$\frac{1}{2}$	$\frac{1}{2}$

$D^{y_b'}_{y_b}$ - Table

Since both y_b and y_b' have the full DC forms

$(A_{i_1-1},\ldots,A_{k+1}, [A^0_k, A_r], A_{r-1},\ldots,A_{i_1+1}, A^0_{i_1}, A^1_k, A^1_{i_1}, A_{k-1},\ldots,A_1)$

at $i"$, $i" = u'(\sum_{h=k+1}^{r} \lambda_h + 1) + \sum_{h=i_1}^{r} \lambda_h$, we see from tables

$D^{y_b}_w$, $D^{y_b'}_w$, $D^{y_b'}_{y_b}$ that $e((y_b), j^1_{A^1_{i_1}}(y_b)) \in \bar{e}((y_b), E_u)$,

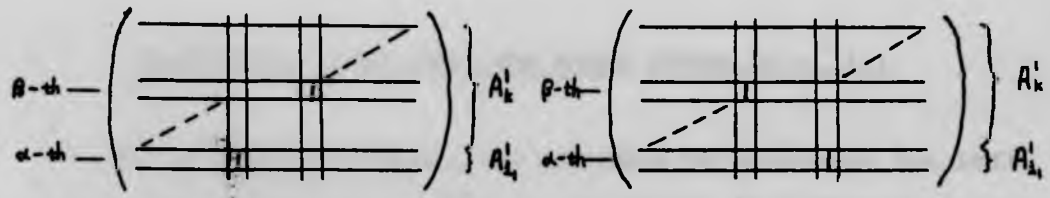
$e((y_b'), j^1_{A^1_{i_1}}(y_b')) \in \bar{e}((y_b'), E_{u-1})$ and y_b' is obtained from y_b

by transposing the α-th and the β-th rows, where $\alpha = i" + \sum_{h=k}^{r} \lambda_h$,

the β-th row of y_b contains an entry belonging to $\bar{e}((y_b), E_{u-1})$

and also to $A^1_k(y_b)$. So by Lemma 9.7.3, y_b, y_b' must have the

forms as below:



Y_b $\qquad\qquad$ Y_b'

So by Lemma 10.1.1, we have $y_b > y_b'$ with $\ell(y_b) = \ell(y_b') + 1$ and hence $y_b \succ y_b'$. According to Theorem B, either $y_a \succ y_a'$ or $y_a \prec y_a'$ holds. Therefore $y_a' \underset{L}{<} y_a$.

On the other hand, let

$$\begin{cases} y_c = \rho^{A_{i_1}^0}_{A_k^1}\left(\rho^{A_{i_1}}_{A_1,\dots,A_{k-1}}\quad \rho^{A_k^0,A_r,\dots,A_{i_1+1}}_{A_1,\dots,A_{k-1},A_k^1}\left(\rho^{A_k^0,A_r,\dots,A_{k+1}}_{A_1,\dots,A_{k-1},A_k^1}\right)^{\alpha_{\lambda,0}-1}\quad \rho^{A_r,\dots,A_{k+1}}_{A_1,\dots,A_{k-1},A_k^1}(w)\right. \\[2ex] \quad = \rho^{A_{i_1}^0}_{A_k^1}\left(\rho^{A_{i_1}}_{A_1,\dots,A_{k-1}}\quad \rho^{A_r,\dots,A_{i_1+1}}_{A_1,\dots,A_{k-1},A_k^1}\left(\rho^{A_r,\dots,A_{k+1},A_k^0}_{A_1,\dots,A_{k-1},A_k^1}\right)^{\alpha_{\lambda,0}-1}\quad (y_a)\right. \\[2ex] y_c' = \rho^{A_{i_1}^0}_{A_k^1}\left(\rho^{A_{i_1}}_{A_1,\dots,A_{k-1}}\quad \rho^{A_r,\dots,A_{i_1+1}}_{A_1,\dots,A_{k-1},A_k^1}\left(\rho^{A_r,\dots,A_{k+1},A_k^0}_{A_1,\dots,A_{k-1},A_k^1}\right)^{\alpha_{\lambda,0}-1}\quad (y_a')\right. \end{cases}$$

Then y_c is the last term of $\xi(w,k)$ but y_c' is not the last term of $\xi(y_a',k)$. By Lemma 9.6.1, we see that $\mathcal{L}(y_c) \not\subseteq \mathcal{L}(y_c')$. We know that y_c' is obtained from y_a' by applying the same sequence of left star operations as are needed to obtain y_c from y_a in $\xi(w,k)$. According to Theorem B, we have either $y_c \prec y_c'$ or $y_c \succ y_c'$. So $y_c \underset{L}{<} y_c'$. Hence $y_a \underset{P_L}{\sim} y_c \underset{L}{<} y_c' \underset{P_L}{\sim} y_a' \underset{L}{<} y_a$. This implies $y_a \underset{L}{\sim} y_a'$. Finally, it follows from $w \underset{P_L}{\sim} y_a$ and $w' \underset{P_L}{\sim} y_a'$ that $w \underset{L}{\sim} w'$.

CHAPTER 11 : THE LEFT AND RIGHT CELLS IN $\sigma^{-1}(\lambda)$

In the present chapter, we are able to determine the left cells of A_n which lie in $\sigma^{-1}(\lambda)$. We recall from Chapter 6 that $\sigma^{-1}(\lambda)$ is a union of left cells. We define in §11.1 a map $T: N_\lambda \to C_\lambda$ from the set of normalized elements of type λ into the set of λ-tabloids, and we shall show that two elements of N_λ lie in the same left cell if and only if they have the same image under T.

In proving this result, it is of crucial importance to identify certain particular left cells in $\sigma^{-1}(\lambda)$. These are obtained from one fixed left cell by applying powers of the automorphism ϕ. This fixed left cell is obtained as follows: We show in §11.3 that there exists a subset J of Δ such that $\{w \in \sigma^{-1}(\lambda) | \mathcal{R}(w) = J\}$ is a left cell.

The result that any two elements x, y lying in the same left cell can be transformed from one to another by a succession of raising operations in N_λ and left star operations will be applied in Chapter 15 to prove the connectness of cells of A_n, where $\lambda = \sigma(x)$.

§11.1 THE MAP T FROM N_λ TO THE SET OF λ-TABLOIDS

Fix $\lambda = \{\lambda_1 \geq \ldots \geq \lambda_r\} \in \Lambda_n$. Let $\mu = \{\mu_1 \geq \ldots \geq \mu_m\}$ be the dual partition of λ.

Recall the definition of a tabloid in §1.2. Now we define a λ-tabloid to be an array of n numbers $\{1,2,\ldots,n\}$ into λ_1 columns such that its u-th column contains μ_u numbers for any u, $1 < u < \lambda$. We say two λ-tabloids are equal if, for any u with $1 < u < \lambda_1$, the u-th columns of these two tabloids contain the same set of numbers. Let C_λ be the set of all λ-tabloids. Then we have

<u>Lemma 11.1.1</u>

$$|C_\lambda| = \frac{n!}{\prod_{j=1}^{m} \mu_j!} .$$

$\square$

For any $w \in \aleph_\lambda$, w has a standard MDC form $(A_r,\ldots,A_1)$ at $i \in \mathbb{Z}$. Let $E_u = \{j_t^u(w) \mid 1 < t < \mu_u\} \subseteq \underline{n}$. Clearly, E_u is independent of the choice of the standard MDC form of w. Let T(w) be the element of C_λ whose u-th column contains all numbers α with $1 < \alpha < n$ and $\bar{\alpha} \in E_u$ for any u, $1 < u < \lambda_1$. Then T(w) is uniquely determined by w. So we can define a map $T: \aleph_\lambda \to C_\lambda$ by sending w to T(w).

<u>Lemma 11.1.2</u> <u>The map T is surjective.</u>

<u>Proof</u> For any $X \in C_\lambda$, assume that the set of numbers in the t-th column of X is

$$E_t = \{i_{th} \mid 1 < h < \mu_t,\ 1 < i_{t\mu_t} < i_{t,\mu_t-1} < \cdots < i_{t1} < n\}$$

for $1 < t < \lambda_1$. Let $M = (M_{\lambda_1}, M_{\lambda_1-1},\ldots,M_1)$ be an $n \times \lambda_1 n$ matrix with M_t being an $n \times n$ matrix, $1 < t < \lambda_1$, such that

(i) for any t, $1 < t < \lambda_1$, $M_t = \begin{pmatrix} M_{tr} \\ M_{t,r-1} \\ \vdots \\ M_{t1} \end{pmatrix}$ with M_{tj} being a

$\lambda_j \times n$ matrix, $1 < j < r$.

(ii) the entries of M_{tj} are all zero if $\mu_t < j < r$; and are all zero except the (t,i_{tj})-entry which is 1 if $1 < j < \mu_t$.

It is clear that there exists a unique element $\tilde{X}(M)$ of $\tilde{A}_n$ with M as a submatrix.

Suppose $i_{\alpha\beta} = 1$ for some α, β with $1 < \alpha < \lambda_1$ and $1 < \beta < \mu_\alpha$. Then there also exists a unique element $X(M)$ of A_n with M as a submatrix and with $(\alpha, i_{\alpha\beta})$-entry of $M_{\alpha\beta}$ lying in its first column, where $M_{\alpha\bar{\beta}}$ is a submatrix of $X(M)$ in M which, regarded as a submatrix of M, is just defined as above.

Clearly, $X(M)$ has the full MDC form $(A_r,\ldots,A_1)$ at i which is normal for some $i \in \mathbb{Z}$, with $|A_h| = \lambda_h$, $1 < h < r$. By Lemma 9.1.1, $X(M)$ is in N_λ. Also clearly, we have $T(X(M)) = X$. This implies that T is surjective. □

If w, w' $\in N_\lambda$ with $w' = L_{E_u}(w)$ for some u, $1 < u < \lambda_1$, where E_u is the subset of $\underline{n}$ such that $\bar{e}((w),E_u)$ is the u-th layer of w, then we have $T(w) = T(w')$. So by Lemma 11.1.2, it is immediate that

<u>Corollary 11.1.3</u>

<u>The map $\bar{T} = T|_{\bar{N}_\lambda} : \bar{N}_\lambda \to C_\lambda$ is surjective.</u> □

§11.2 <u>THE SET $\bar{N}_\lambda$ OF PRINCIPAL NORMALIZED ELEMENTS</u>

We shall show in this section that for any given $X \in C_\lambda$ and any number α in the first column of X, there exists a unique element w of $\bar{N}_\lambda$ such that w has a standard MDC form $(A_r,\ldots,A_1)$

at i for some $i \in \mathbb{Z}$ with $\overline{j_1^1(w)} = \bar{\alpha}$ and $T(w) = X$. By using this result, we can precisely calculate the cardinal of $\bar{T}^{-1}(X)$ for any $X \in C_\lambda$ and then the cardinal of $\bar{N}_\lambda$.

The main purpose of this section is to show that for any $X \in C_\lambda$, $T^{-1}(X)$ lies in some left cell of A_n.

Suppose that $E = \{\bar{i}_1, \dots, \bar{i}_t\} \subseteq \underline{n}$ is a non-empty set with $1 \leq i_1 < \dots < i_t \leq n$. Let $\bar{e}((w),E)$ be the set of entry classes of $w \in A_n$ such that $\bar{e}((w),E)$ contains all entries $e(i,j)$ with $\bar{j} \in E$. Let $\bar{e}((w),j_1)$, $e((w),j_2)$ be two entries of w lying in $\bar{e}((w),E)$ with $j_1 < j_2$ such that there exists no entry $e((w),j)$ of w lying in $\bar{e}((w),E)$ with $j_1 < j < j_2$. Suppose $\bar{j}_1 = \bar{i}_\alpha$ (resp. $\bar{j}_2 = \bar{i}_\alpha$) for some α, $1 \leq \alpha \leq t$. This implies $\bar{j}_2 = \bar{i}_{\alpha+1}$ (resp. $\bar{j}_1 = \overline{i_{\alpha-1}}$) and $0 < j_2 - j_1 \leq n$ with the convention that $i_\beta = i_{\beta+qt}$ for any β, q with $1 \leq \beta \leq t$ and $q \in \mathbb{Z}$.

Now suppose $\lambda = \{\lambda_1 \geq \dots \geq \lambda_r\} \in \Lambda_n$ and $w \in N_\lambda$ has a standard MDC form $(A_r, \dots, A_1)$ at i for some $i \in \mathbb{Z}$ with $T(w) = X \in C_\lambda$. Let X_k be the set of all numbers in the k-th column of X, $1 \leq k \leq \lambda_1$. Then $\bar{X}_k = \{\overline{j_t^k(w)} \mid 1 \leq t \leq \mu_k\}$. Since $\bar{e}((w),E_k)$, $1 \leq k \leq \lambda_1$, are increasing chains of w, we see by the above statement that for any k, $1 \leq k \leq \lambda_1$, once one number $j_t^k(w)$ is known for some t, $1 \leq t \leq \mu_k$, all the numbers $j_v^k(w)$, $1 \leq v \leq \mu_k$, will be determined.

We say an increasing sequence $E: i_1, \dots, i_t$ of $\mathbb{Z}$ is compact with respect to n, $n \geq 1$, if inequality $i_t - n < i_1 < \dots < i_t$ holds. (Subsequently, the number n is always fixed. So we may just say E is compact without danger of confusion.)

Clearly, if $E: i_1, \ldots, i_t$ and $F: j_1, \ldots, j_t$ are two compact increasing sequences of $\mathbb{Z}$ such that $\bar{E}, \bar{F} \subseteq \underline{n}$ with $\bar{E} = \bar{F}$ and $i_1 = j_1$, then we have $i_\alpha = j_\alpha$ for any α, $1 \le \alpha \le t$.

Suppose $E: i_1, \ldots, i_t$ and $F: j_1, \ldots, j_t$ are two compact increasing sequences. We say E dominates F if the following conditions are satisfied:

(i) $\bar{E}, \bar{F} \subset \underline{n}$ with $\bar{E} \cap \bar{F} = \emptyset$.

(ii) $i_\alpha > j_\alpha$ for any α, $1 \le \alpha \le t$.

Suppose E dominates F. We say E, F are interlocking if the following additional condition is also satisfied:

(iii) There exists β, $1 \le \beta \le t$, such that $i_{\beta-1} < j_\beta$ with the convention that $i_a = i_{a+qt} - qn$ and $j_b = j_{b+qt} - qn$ for any $a, b, q \in \mathbb{Z}$ with $1 \le a, b \le t$.

<u>Example</u>: Suppose $w \in \mathbb{N}_\lambda$ has a standard MDC form $(A_r, \ldots, A_1)$ at $i \in \mathbb{Z}$. For any k, h with $1 \le k \le \lambda_1$ and $1 \le h < \lambda_1$, let $E_k: j^k_{\mu_k}(w), j^k_{\mu_k-1}(w), \ldots, j^k_1(w)$ and let $E'_h: j^h_{\mu_{h+1}}(w), j^h_{\mu_{h+1}-1}(w), \ldots$ $\ldots, j^h_1(w)$. Then by definition of $\mathbb{N}_\lambda$, we see that E_k, E'_h are all compact increasing sequences of $\mathbb{Z}$ and that E'_h dominates E_{h+1} for any h, $1 \le h < \lambda_1$. Moreover, w is in $\bar{\mathbb{N}}_\lambda$ if and only if E'_h, E_{h+1} are interlocking for any h, $1 \le h < \lambda_1$.

<u>Lemma 11.2.1</u> <u>Suppose $E: i_1, \ldots, i_t$ is a compact increasing sequence of $\mathbb{Z}$. Suppose there exists $D \subset \underline{n}$ satisfies $|D| = t$ and $D \cap \bar{E} = \emptyset$ in $\underline{n}$. Then there exists a unique compact increasing</u>

<u>sequence $F:j_1,\ldots,j_t$ such that $\bar{\bar{F}} = D$, E dominates F, and E,F are interlocking.</u>

<u>Proof</u> Obviously, there exists a compact increasing sequence $F_1:j_{11},\ldots,j_{1t}$ such that $\bar{\bar{F}}_1 = D$ and E dominates F_1. If E, F_1 are not interlocking, then for any β, $1 \le \beta \le t$, we have $i_{\beta-1} > j_{1\beta}$. Let $F_2:j_{21},\ldots,j_{2t}$ such that $j_{2\alpha} = j_{1,\alpha+1}$ for $1 \le \alpha \le t$ with the convention that $j_{1\beta} = j_{1,\beta+qt} - qn$ for any β, $q \in \mathbb{Z}$ with $1 \le \beta \le t$. Then $\bar{\bar{F}}_2 = D$ and E dominates F_2. If E, F_2 are still not interlocking, then the same procedure can be carried on and we get $F_1, F_2, \ldots$ such that for every $c \ge 1$, the compact increasing sequence $F_c:j_{c1},\ldots,j_{ct}$ satisfies $j_{c\alpha} = j_{1,\alpha+c-1}$ for all α, $1 \le \alpha \le t$. It is clear that $\bar{\bar{F}}_c = D$ for $c \ge 1$. If E, F_{b-1} are not interlocking, then E dominates F_b for any $b > 1$. We claim that there exists some $c \ge 1$ such that E, F_c are interlocking. Otherwise, E dominates F_c for any $c \ge 1$. This implies $i_\alpha > j_{c\alpha} = j_{1,\alpha+c-1}$ for any $c \ge 1$ and any α with $1 \le \alpha \le t$. In particular, $i_1 > j_{1,\alpha+qt}$ for any $q > 0$. But i_1 is finite and $\lim_{q\to\infty} j_{1,\alpha+qt} = \lim_{q\to\infty} (j_{1\alpha} + qn) = \infty$, this gives a contradiction. Thus the existence of F is proved.

Now we shall show that such a sequence F is unique. Suppose $F:j_1,\ldots,j_t$ and $F':j_1',\ldots,j_t'$ are two compact increasing sequences of $\mathbb{Z}$ such that $\bar{\bar{F}} = \bar{\bar{F}}' = D$ and E is dominantly interlocking with both F and F'. We may assume $j_1 \le j_1'$. If $j_1 \lneq j_1'$, then there exists some $c \ge 1$ such that $j_1' = j_{1+c}$ and hence $j_\alpha' = j_{\alpha+c}$ for all α, $1 \le \alpha \le t$ with the convention that $j_\alpha = j_{\alpha+qt} - qn$ for any α, q with $1 < \alpha < t$ and $q \in \mathbb{Z}$. Since E dominates F', we have $i_{\beta-1} > j'_{\beta-1} = j_{\beta+c-1} > j_\beta$ for any β, $1 < \beta < t$. This contradicts the fact that E, F are interlocking. Hence we must have $j_1 = j'_1$ and then $j_\alpha = j'_\alpha$ for all α, $1 < \alpha < t$. This proves the uniqueness of F. □

Given $\lambda = \{\lambda_1 \geq \dots \geq \lambda_r\} \in \Lambda_n$ and $X \in C_\lambda$. Let X_j be the set of all numbers in the j-th column of X, $1 < j < \lambda_1$. Then for any $\alpha \in X_1$, we denote the following conditions on an element w of $\mathbb{N}_\lambda$ by $A(X,\alpha)$:

(i) $T(w) = X$

(ii) w has a standard MDC form $(A_r,\dots,A_1)$ at i for some $i \in \mathbb{Z}$ with $\overline{j_1^1(w)} = \bar{\alpha}$.

<u>Lemma 11.2.2</u> <u>Let $\lambda \in \Lambda_n$, $X \in C_\lambda$ and $\alpha \in X_1$ be defined as above. Then there exists an element $w \in \mathbb{N}_\lambda$ which satisfies $A(X,\alpha)$.</u>

<u>Proof</u> We know from Lemma 11.1.2 that there exists an element $y \in \mathbb{N}_\lambda$ which has a standard MDC form $(A_r,\dots,A_1)$ at i for some $i \in \mathbb{Z}$ with $T(y) = X$. Suppose that E is the sequence $j_r^1(y), j_{r-1}^1(y),\dots,j_1^1(y)$. Then E is a compact increasing sequence of $\mathbb{Z}$ with $\bar{E} = \bar{X}_1$. Suppose $\overline{j_k^1(y)} = \bar{\alpha}$ for some k, $1 < k < r$. Then there exists an element $w = (\rho^{A_r}_{A_1,\dots,A_{r-1}})^{r+1-k}(y)$ of A_n by Lemma 5.3.5. Hence we see from Corollary 5.4.7 that w is in $\mathbb{N}_\lambda$ and has a standard MDC form $(A_r,\dots,A_1)$ at $i + (r+1-k)\lambda_r$ with $T(w) = X$ and $\overline{j_1^1(w)} = \overline{j_k^1(y)} = \bar{\alpha}$. So w is as required. □

<u>Lemma 11.2.3 Assume that $w \in \bar{N}_\lambda$ has a standard MDC form $(A_r,\dots,A_1)$ at i for some $i \in \mathbb{Z}$. Assume that $w' \in \bar{N}_\lambda$ satisfies $w' = L_{E_u}(w)$ for some u, $1 < u < \lambda_1$. Then w satisfies $A(X,\alpha)$ if and only if w' satisfies $A(X,\alpha)$, where $\lambda \in \Lambda_n$, $X \in C_\lambda$ and $\alpha \in X_1$ are as in Lemma 11.2.2.</u>

<u>Proof</u> This is obvious. □

<u>Proposition 11.2.4 Let $\lambda \in \Lambda_n$, $X \in C_\lambda$ and $\alpha \in X_1$ be as in Lemma 11.2.2. Then there exists a unique element w in $\bar{N}_\lambda$ which satisfies $A(X,\alpha)$</u>.

<u>Proof</u> The existence of w follows from Lemmas 11.2.2, 11.2.3 and Proposition 10.2.4.

Now we shall show the uniqueness of w. Assume that $w \in \bar{N}_\lambda$ satisfies $A(X,\alpha)$ and has a standard MDC form $(A_r,\dots,A_1)$ at i for some $i \in \mathbb{Z}$. We may assume $1 < j_1^1(w) < n$ in this form. Then the number $j_1^1(w)$ is uniquely determined by equation $\overline{j_1^1(w)} = \bar{\alpha}$.

Let $E_t: j_{\mu_t}^t(w),\dots,j_1^t(w)$ for any t, $1 < t < \lambda_1$. Then E_t are all compact increasing sequences of $\mathbb{Z}$ with $\bar{E}_t = \bar{X}_t$. Since $\overline{j_1^1(w)} = \bar{\alpha}$ and $\bar{E}_1 = \bar{X}_1$, the sequence $j_r^1(w),\dots,j_1^1(w)$ is uniquely determined by $j_1^1(w)$. In particular, the sequence $E_1': j_{\mu_2}^1(w),\dots,j_1^1(w)$ is uniquely determined. It is obvious that E_1' is also a compact increasing sequence of $\mathbb{Z}$ such that E_1', E_2 are interlocking. Since $\bar{E}_2 = \bar{X}_2 \subset \underline{n}$ satisfies $|\bar{E}_2| = |E_1'|$ and

$\bar{E}_2 \cap \bar{E}_1' = \emptyset$ in $\underline{n}$, we see by Lemma 11.2.1 that the sequence E_2 is uniquely determined by E_1'. In general, assume that for some k, $1 < k < \lambda_1$, the sequences E_h, $1 < h < k$, are all uniquely determined. We know that $E_k': j^k_{\mu_{k+1}}(w), \ldots, j^k_1(w)$ and E_{k+1} are two compact increasing sequences of $\mathbb{Z}$ with E_k', E_{k+1} interlocking. We also know that the sequence E_k' and the set $\bar{E}_{k+1} = \bar{X}_{k+1}$ have been determined. This implies by Lemma 11.2.1 that the sequence E_{k+1} must be uniquely determined. By applying induction on k, we see that for any k, h with $1 < k < \lambda_1$ and $1 < h < \mu_k$, $j^k_h(w)$ is uniquely determined. Then by equation

$$\sum_{\substack{1<k<\lambda_1 \\ 1<h<\mu_k}} j^k_h(w) = in + \sum_{h=1}^{n} h$$

we can get a unique integer solution for i. This means that w is uniquely determined by condition $A(X,\alpha)$. Our proof is complete. □

Now we can prove the main result in this section.

<u>Proposition 11.2.5</u> <u>For any $X \in C_\lambda$, the fibre $T^{-1}(X)$ belongs to some left cell of $\tilde{A}_n$.</u>

<u>Proof</u> Take any number α in the first column of X. By Proposition 11.2.4, there exists a unique element w in $\bar{N}_\lambda$ which satisfies $A(X,\alpha)$.

Now let y be any element of $T^{-1}(X)$. By the proof of Lemma 11.2.2, we see that there exists $x \in N_\lambda$ satisfying $x \underset{F_L}{\sim} y$ and

$A(X,\alpha)$. Then by Lemmas 11.2.3, 10.2.2 and Proposition 10.2.3, we can transform x by a succession of raising operations on the u-th layers of w with $1 < u < \lambda_1$ and get an element of $\bar{N}_\lambda$ which is in the same left cell as x and satisfies $A(X,\alpha)$. Thus by Proposition 11.2.4, this element must be w. Hence we have $y \underset{L}{\sim} w$. Thus any element of $T^{-1}(X)$ lies in the same left cell as w. Our conclusion follows. □

For any $X \in C_\lambda$, let $y, y' \in \bar{T}^{-1}(X)$ be such that y (resp. y') has a standard MDC form $(A_r,\ldots,A_1)$ at i (resp. i'). Let X_1 be the set of numbers in the first column of X. Then by Proposition 11.2.4, we see that $\overline{j_1^1(y)} = \overline{j_1^1(y')}$ implies $y = y'$. If $\lambda_1 = \lambda_r$, then by proper choice of standard MDC forms of y, y', we can always make $\overline{j_1^1(y)} = \overline{j_1^1(y')}$. So in that case, the cardinal of $\bar{T}^{-1}(X)$ is 1. But if $\lambda_1 \gneq \lambda_r$, then the residue class $\overline{j_1^1(y)}$ (resp. $\overline{j_1^1(y')}$) is independent of the choice of the standard MDC form of y (resp. y'). Clearly, when $\overline{j_1^1(y)} \neq \overline{j_1^1(y')}$, we have $y \neq y'$. So the cardinal of $\bar{T}^{-1}(X)$ must be r.

<u>Proposition 11.2.6</u> <u>Assume $\lambda = \{\lambda_1 > \ldots > \lambda_r\} \in \Lambda_n$. Then for any $X \in C_\lambda$, $|\bar{T}^{-1}(X)|$ is a constant: it is equal to 1 if $\lambda_1 = \lambda_r$, or equal to r if $\lambda_1 \gneq \lambda_r$. So $|\bar{N}_\lambda|$ is equal to $\frac{n!}{\prod_{j=1}^{m} \mu_j!}$ if $\lambda_1 = \lambda_r$ or equal to $\frac{r(n!)}{\prod_{j=1}^{m} \mu_j!}$ if $\lambda_1 \gneq \lambda_r$, where $\mu = \{\mu_1 > \ldots > \mu_m\}$ is the dual partition of λ.</u>

Proof By the above statement and Lemmas 11.1.1, 11.1.2. □

§11.3 THE SUBSET X_λ OF N_λ

For any v, t with $0 \leq v < n$ and $1 \leq t \leq \lambda_1$, let

$E_{vt} = \{v + \sum_{h=j}^{r} \lambda_h + 1 - t \mid 1 \leq j \leq \mu_t\}$. Let X^v be a tabloid of C_λ with the set of numbers modulo n in its t-th column to be E_{vt}. Let $X_\lambda = \bigcup_{v=0}^{n-1} T^{-1}(X^v)$.

In this section, we shall show $X_\lambda = \{\omega \in N_\lambda \mid w^{-1} \in H_\lambda\}$ and then show that for any $x,y \in \sigma^{-1}(\lambda)$ with $x^{-1} \in H_\lambda$, $x \underset{L}{\sim} y$ if and only if $R(x) = R(y)$.

For any v with $0 \leq v < n$, let $J_\lambda(v) = J_{r,v} \cup \ldots \cup J_{1,v}$, where $J_{h,v} = \{s_{\alpha_h+1}, s_{\alpha_h+2}, \ldots, s_{\alpha_h+\lambda_h-1}\}$, $\alpha_h = v + \sum_{\ell=h+1}^{r} \lambda_\ell$. Then by the latter part of Chapter 3, we see that $w_o^{J_\lambda(v)}$ is in $T^{-1}(X^v)$. So by Proposition 11.2.5, it is immediate that

Lemma 11.3.1 *Any $w \in T^{-1}(X^v)$ is in the same left cell as $w_o^{J_\lambda(v)}$ for any v, $0 \leq v < n$.* □

Let $w \in N_\lambda$ have a standard MDC form $(A_r,\ldots,A_1)$ at $i \in Z$. Let $\bar{e}((w),E_u)$ be the u-th layer of w for any u, $1 \leq u \leq \lambda_1$.

Lemma 11.3.2 *Assume that $e(i_t,j_t)$ are in $\bar{e}((w),E_{u_t})$, t = 1,2, with $i_1 < i_2$ and $j_1 > j_2$. Then $u_1 < u_2$.*

<u>Proof</u> By Lemma 3.10, we know $u_1 \neq u_2$. Suppose $u_1 > u_2$. We may assume without loss of generality that $j_1 = j_h^{u_1}(w)$ for some h, $1 \le h \le \mu_1$. Let $e(i_3,j_3)$ be $e((w), j_h^{u_2}(w))$. Then $i_3 < i_1$ and $j_3 = j_h^{u_2}(w) > j_h^{u_1}(w) = j_1$. So $e(i_3,j_3)$ is not $e(i_2,j_2)$. Since $\bar{e}((w),E_{u_2})$ is an increasing chain of w, we have $(i_2-i_3)(j_2-j_3) > 0$. But on the other hand, $i_2 - i_3 = (i_2-i_1) + (i_1-i_3) > 0$ and $j_2-j_3 = (j_2-j_1) + (j_1-j_3) < 0$. This implies $(i_2-i_3)(j_2-j_3) < 0$, a contradiction. Therefore $u_1 < u_2$. □

If $w \in \mathbb{M}_\lambda$ with $w^{-1} \in H_\lambda$, then there exists $i' \in \mathbb{Z}$ such that $\{e(i_t^v(w),\ i' + \sum_{h=t}^{r} \lambda_h + 1-v) \mid 1 \le v \le \lambda_t\}$ is a descending chain of w for all t, $1 \le t \le r$. Assume that the entry $e(i_t^v(w),\ i' + \sum_{h=t}^{r} \lambda_h + 1-v)$ is in $\bar{e}((w),E_{\alpha_{tv}})$. Then by Lemma 11.3.2, we have $\alpha_{t1} < \alpha_{t2} < \dots < \alpha_{t\lambda_t}$ for $1 \le t \le r$. Since

$\{i' + \sum_{h=t}^{r} \lambda_h + 1-v \mid 1 \le t \le r,\ 1 \le v \le \lambda_t\} = \underline{n}$, by successively

considering the sets $\{\alpha_{tv} \mid 1 \le v \le \lambda_t\}$ where t runs from 1 to r, we can see that $\alpha_{tv} = v$ for any t, $1 \le t \le \mu_v$. Therefore $E_v = \{i' + \sum_{h=t}^{r} \lambda_h + 1-v \mid 1 \le t \le \mu_v\}$ for all v, $1 \le v \le \lambda_1$, and we have $w \in X_\lambda$.

<u>Proposition 11.3.3</u> $\underline{X_\lambda = \{w \in \mathbb{M}_\lambda \mid w^{-1} \in H_\lambda\}}$.

<u>Proof</u> We have shown $\{w \in \mathbb{M}_\lambda \mid w^{-1} \in H_\lambda\} \subseteq X_\lambda$. By Theorem A(i), we know $\mathcal{R}(x) = \mathcal{R}(y)$ for any x,y with $x \underset{L}{\sim} y$. So Lemma 11.3.1 tells us that for any $w \in X_\lambda$, $\mathcal{R}(w) = J_\lambda(v)$ with some v, $0 < v < n$.

i.e. $w^{-1} \in H_\lambda$. This implies $X_\lambda \subseteq \{w \in N_\lambda | w^{-1} \in H_\lambda\}$. Our result follows. □

Proposition 11.3.4 Assume $x, y \in \sigma^{-1}(\lambda)$ with $x^{-1} \in H_\lambda$. Then $x \underset{L}{\sim} y$ if and only if $\mathcal{R}(x) = \mathcal{R}(y)$.

Proof ($\Rightarrow$) By Theorem A(i).

($\Leftarrow$) By Proposition 10.2.5, there exist $x', y' \in N_\lambda$ such that $x' \underset{L}{\sim} x$ and $y' \underset{L}{\sim} y$. Then condition $\mathcal{R}(x) = \mathcal{R}(y)$ implies $\mathcal{R}(x') = \mathcal{R}(y') = \mathcal{R}(x)$ by Theorem A(i). Since $x^{-1} \in H_\lambda$, it follows that $x'^{-1}, y'^{-1} \in H_\lambda$. Hence Proposition 11.3.3 together with $\mathcal{R}(x') = \mathcal{R}(y')$ implies that $T(x') = T(y') = X^v$ for some v, $0 < v < n$. So by Proposition 11.2.5, we get $x' \underset{L}{\sim} y'$ and then $x \underset{L}{\sim} y$. □

§11.4 THE NUMBER OF LEFT CELLS IN $\sigma^{-1}(\lambda)$

So far, we have shown that for any $X \in C_\lambda$, the fibre $T^{-1}(X)$ lies in some left cell by using the properties of a principal normalized element. In any principal normalized element, we see that the distance between any two adjacent layers is very short. Now we shall define a new kind of set which consists of all elements w of N_λ such that the distance between any two adjacent layers of w goes beyond some fixed bound. By using the properties of these elements, we shall find all the left cells of $\sigma^{-1}(\lambda)$.

Assume that $w \in \overline{N_\lambda}$ has a standard MDC form $(A_r,\dots,A_1)$ at $i \in \mathbb{Z}$ and $E_u = \{j_h^u(w) \mid 1 \le h \le \mu_u\}$, $1 \le u \le \lambda_1$. For $\ell > 0$, we say that $x = (L_{E_u})^{\ell}(w)$ is available from w in N_λ if $(L_{E_u})^{j}(w) \in N_\lambda$ for any j, $1 \le j \le \ell$. We say that $x = (L_{E_{u_1}})^{\ell_1}(L_{E_{u_2}})^{\ell_2}\dots(L_{E_{u_t}})^{\ell_t}(w)$ is available from w in N_λ if $x_h = (L_{E_{u_h}})^{\ell_h}((L_{E_{u_{h+1}}})^{\ell_{h+1}} \dots (L_{E_{u_t}})^{\ell_t}(w))$ is available from $(L_{E_{u_{h+1}}})^{\ell_{h+1}} \dots (L_{E_{u_t}})^{\ell_t}(w)$ in N_λ for any h, $1 \le h \le t$.

Suppose that $w_1 = (L_{E_v})^{\mu_v}(w)$ is available from w in N_λ, then w_1 has a standard MDC form $(A_r,\dots,A_1)$ at i', $i' = i + \mu_v$, and for any t, $1 \le t \le r$,

$$j_t^u(w_1) = \begin{cases} j_t^u(w) & \text{if } 1 \le u \le \lambda_t,\ u \ne v \\ j_t^v(w)+n & \text{if } u = v. \end{cases}$$

By this formula, it follows that if $w_1 = (L_{E_v})^{m\mu_v}(w)$ is available from w in N_λ, $1 \le v \le \lambda_1$, $m > 0$, then $w_2 = (L_{E_{v+1}})^{\ell}(w_1)$ is available from w_1 in N_λ for any ℓ, $1 \le \ell \le m\mu_{v+1}$. In particular, $(L_{E_1})^{m}(w)$ is always available from w in N_λ for any $m > 0$. Given any integer $N > 0$, let m_α, $1 \le \alpha \le \lambda_1$, be a set of integers such that $m_1 > \dots > m_{\lambda_1} > 0$ with $m_j - m_{j+1} > N + 2$, $1 \le j \le \lambda_1$ then $w' = (L_{E_{\lambda_1}})^{m_{\lambda_1}\mu_{\lambda_1}}(L_{E_{\lambda_1-1}})^{m_{\lambda_1-1}\mu_{\lambda_1-1}}\dots(L_{E_1})^{m_1\mu_1}(w)$ is available from w in N_λ such that

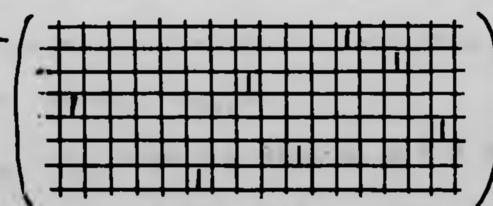
$$\begin{cases} j^u_t(w')-j^{u'}_{t'}(w') > Nn, \text{ for any } t,t',u,u' \text{ with } u < u', \\ \qquad\qquad 1 \le t \le \mu_u \text{ and } 1 \le t' \le \mu_{u'}, \\ j^u_1(w')-n < j^u_{\mu_u}(w') < j^u_{\mu_u-1}(w') < \dots < j^u_1(w'), \text{ for } 1 \le u \le \lambda_1. \end{cases}$$

By Proposition 10.2.3, we see that $w' \in \mathbb{N}_\lambda$ satisfies $w' \underset{L}{\sim} w$.

For $a \in \mathbb{Z}$, we define $\mathbb{N}_\lambda(a)$ to be the set of all elements w of $\mathbb{N}_\lambda$ such that w has a standard MDC form $(A_r,\dots,A_1)$ at $i \in \mathbb{Z}$ satisfying $j^u_t(w) - j^{u'}_{t'}(w) > a$ for any t,t',u,u' with $u < u'$, $1 \le t \le \mu_u$ and $1 \le t' \le \mu_{u'}$.

For example, suppose $n = 7$, $\lambda = \{3 \ge 3 \ge 1\}$. Then the following element w is in $\mathbb{N}_\lambda(1)$ but not in $\mathbb{N}_\lambda(2)$.

(i+1)-th



w

It is clear that $\mathbb{N}_\lambda(-n) = \mathbb{N}_\lambda$ and $\mathbb{N}_\lambda(a) \subseteq \mathbb{N}_\lambda(a')$ for any $a, a' \in \mathbb{Z}$ with $a \ge a'$.

By the above discussion, for any $a \in \mathbb{Z}$, $w \in \mathbb{N}_\lambda$, there always exists an element w' in $\mathbb{N}_\lambda(a)$ such that $w' \underset{L}{\sim} w$.

<u>Lemma 11.4.1</u> <u>For $a \ge n$, let $w \in \mathbb{N}_\lambda(a)$ have a standard MDC form $(A_r,\dots,A_1)$ at $i \in \mathbb{Z}$. Let $e(i_t,t)$ be in the layer $\bar{e}((w),E_{\alpha_t})$ and $e(i_{t+\varepsilon},t+\varepsilon)$ in the layer $\bar{e}((w),E_{\alpha_{t+\varepsilon}})$, $\varepsilon = 1,2$.</u>

Then (i) $\alpha_t = \alpha_{t+\varepsilon} \Leftrightarrow 0 < i_{t+\varepsilon} - i_t < n$

(ii) $\alpha_t < \alpha_{t+\varepsilon} \Leftrightarrow i_{t+\varepsilon} - i_t > n$

(iii) $\alpha_t > \alpha_{t+\varepsilon} \Leftrightarrow i_{t+\varepsilon} - i_t < 0$

Proof Since $\bar{t} \neq \overline{t+\varepsilon}$, we have $\bar{i}_t \neq \overline{i_{t+\varepsilon}}$. So $i_{t+\varepsilon} - i_t \neq 0,n$. We may assume that $e(i_t,t)$ is $e((w),j_h^{\alpha_t}(w))$ for some h, $1 \leq h \leq \mu_{\alpha_t}$. We need only show the implication in the direction "$\Leftarrow$" for all these three cases.

(i) Either $e(i_{t+\varepsilon}, t+\varepsilon)$ or $e(i_{t+\varepsilon} -n, t+\varepsilon-n)$ is $e((w),j_{h'}^{\alpha_{t+\varepsilon}}(w))$ for some h', $1 \leq h' \leq \mu_{\alpha_{t+\varepsilon}}$. If $\alpha_t \neq \alpha_{t+\varepsilon}$, then $|j_h^{\alpha_t}(w) - j_{h'}^{\alpha_{t+\varepsilon}}(w)| > a \geq n$. But in fact, $|j_h^{\alpha_t}(w)-j_{h'}^{\alpha_{t+\varepsilon}}(w)| = |t-(t+\varepsilon)|$ or $|t-(t+\varepsilon-n)|$ which is less than n. This is a contradiction. So $\alpha_t = \alpha_{t+\varepsilon}$.

(ii) Let $i_{t+\varepsilon} - i_t = qn + p$ with q, $p \in \mathbb{Z}$ and $0 < p < n$. Then $q > 0$. Let $e(i_o,j_o)$ be $e(i_{t+\varepsilon} - qn, t+\varepsilon-qn)$. Then $e(i_o,j_o)$ is in $\bar{e}((w),E_{\alpha_{t+\varepsilon}})$. Since $i_t < i_o$ and $t > j_o$, this implies $\alpha_t < \alpha_{t+\varepsilon}$ by Lemma 11.3.2

(iii) By Lemma 11.3.2. □

Let w be as in Lemma 11.4.1. Let $e(i_\beta(w),\beta)$ be in $\bar{e}((w),E_{\alpha_\beta(w)})$; for $\beta = t, t+1, t+2$ with $t \in \mathbb{Z}$. Then $w \in D_R(S_t)$ if and only if one of the following cases occurs by §2.3.

(i) $i_{t+1}(w) < i_t(w) < i_{t+2}(w)$, (ii) $i_{t+1}(w) < i_{t+2}(w) < i_t(w)$;

(iii) $i_t(w) < i_{t+2}(w) < i_{t+1}(w)$: (iv) $i_{t+2}(w) < i_t(w) < i_{t+1}(w)$.

By Lemma 11.4.1, we have

inequality (i) holds $\Longleftrightarrow \alpha_{t+1}(w) < \alpha_t(w) < \alpha_{t+2}(w)$

inequality (ii) holds $\Longleftrightarrow \alpha_{t+1}(w) < \alpha_{t+2}(w) < \alpha_t(w)$

inequality (iii) holds $\Longleftrightarrow \alpha_t(w) < \alpha_{t+2}(w) < \alpha_{t+1}(w)$

inequality (iv) holds $\Longleftrightarrow \alpha_{t+2}(w) < \alpha_t(w) < \alpha_{t+1}(w)$.

When $w \in \mathcal{D}_R(s_t)$, let $w' = w^*$ in $\mathcal{D}_R(s_t)$. Let $e(i_\beta(w'),\beta)$ be the entry of w', $\beta = t, t+1, t+2$. Then we have

$(i_{t+1}(w'), i_t(w'), i_{t+2}(w')) = (i_{t+2}(w), i_t(w), i_{t+1}(w))$ in case (i)

$(i_{t+1}(w'), i_{t+2}(w'), i_t(w')) = (i_t(w), i_{t+2}(w), i_{t+1}(w))$ in case (ii)

$(i_t(w'), i_{t+2}(w'), i_{t+1}(w')) = (i_{t+1}(w), i_{t+2}(w), i_t(w))$ in case (iii)

$(i_{t+2}(w'), i_t(w'), i_{t+1}(w')) = (i_{t+1}(w), i_t(w), i_{t+2}(w))$ in case (iv).

Since w' is obtained from w by transposing two consecutive columns of w which contain the entries lying in distinct layers of w, we see that w' lies in $\mathcal{H}_\lambda(a-2)$ which also has a standard MDC form $(A_r,\ldots,A_1)$ at i.

Assume that $e(i_\beta(w'),\beta)$ is in $\bar{e}((w'), E_{\alpha_\beta(w')})$. Then we have

$(\alpha_{t+1}(w'), \alpha_t(w'), \alpha_{t+2}(w')) = (\alpha_{t+2}(w), \alpha_t(w), \alpha_{t+1}(w))$ in case (i)

$(\alpha_{t+1}(w'), \alpha_{t+2}(w'), \alpha_t(w')) = (\alpha_t(w), \alpha_{t+2}(w), \alpha_{t+1}(w))$ in case (ii)

$(\alpha_t(w'), \alpha_{t+2}(w'), \alpha_{t+1}(w')) = (\alpha_{t+1}(w), \alpha_{t+2}(w), \alpha_t(w))$ in case (iii)

$(\alpha_{t+2}(w'), \alpha_t(w'), \alpha_{t+1}(w')) = (\alpha_{t+1}(w), \alpha_t(w), \alpha_{t+2}(w))$ in case (iv).

So in other words, we can say that $w \in \mathcal{D}_R(s_t)$ if and only if one of the following cases occurs:

(i') $\alpha_t(w)$, $\alpha_{t+1}(w)$ and $\alpha_{t+2}(w)$ are distinct, one of $\alpha_t(w)$ and $\alpha_{t+2}(w)$, say $\alpha(w)$, lies between the others.

(ii') Two of $\alpha_t(w)$, $\alpha_{t+1}(w)$, $\alpha_{t+2}(w)$ are equal, but the remaining one, say $\alpha(w)$, is distinct. When $\alpha(w) \neq \alpha_{t+1}(w)$, then either $\alpha(w) = \alpha_t(w)$ is maximum or $\alpha(w) = \alpha_{t+2}(w)$ is minimum among $\{\alpha_\beta(w) \mid \beta = t,t+1,t+2\}$.

In case (i'), let $\alpha'(x) = \{\alpha_t(x), \alpha_{t+2}(x)\} - \{\alpha(x)\}$ for $x = w, w'$. Then $(\alpha_{t+1}(w'),\alpha(w'),\alpha'(w')) = (\alpha'(w),\alpha(w),\alpha_{t+1}(w))$. In case (ii'), when $\alpha(w) \neq \alpha_{t+1}(w)$, let $\alpha'(x) = \{\alpha_t(x), \alpha_{t+2}(x)\} - \{\alpha(x)\}$. Then $(\alpha_{t+1}(w'), \alpha'(w'), \alpha(w')) = (\alpha(w), \alpha'(w), \alpha_{t+1}(w))$.

When $\alpha(w) = \alpha_{t+1}(w)$, then

$(\alpha_{t+1}(w'),\alpha_t(w'),\alpha_{t+2}(w')) = (\alpha_t(w),\alpha_{t+1}(w),\alpha_{t+2}(w))$ if $\alpha_{t+1}(w)$ is maximum.

$(\alpha_{t+1}(w'),\alpha_t(w'),\alpha_{t+2}(w')) = (\alpha_{t+2}(w),\alpha_t(w),\alpha_{t+1}(w))$ if $\alpha_{t+1}(w)$ is minimum.

Therefore we have

<u>Lemma 11.4.2</u> <u>For $w \in \mathcal{N}_\lambda(a)$, $a > n$ and $t \in \mathbb{Z}$, whether w is in $\mathcal{D}_R(s_t)$ or not is entirely determined by $T(w)$. That is, for any $X \in C_\lambda$ and $w, y \in \mathcal{N}_\lambda(a) \cap T^{-1}(X)$, we have that w is in $\mathcal{D}_R(s_t)$ if and only if y is in $\mathcal{D}_R(s_t)$. When they are in $\mathcal{D}_R(s_t)$, let $w' = w^*$, $y' = y^*$ in $\mathcal{D}_R(s_t)$. Then $w', y' \in \mathcal{N}_\lambda(a-2)$ and $T(w') = T(y')$.</u> □

Now we can prove the following result.

<u>Proposition 11.4.3</u> <u>For $x,y \in \mathbb{N}_\lambda$, we have $x \underset{L}{\sim} y \Leftrightarrow T(x) = T(y)$</u>

<u>Proof</u> ($\Leftarrow$) By Proposition 11.2.5

($\Rightarrow$) Otherwise, for any $a,b \in \mathbb{Z}$ with $a > b > 0$, there exists x,y with $x \underset{L}{\sim} y$ such that for some $X, Y \in C_\lambda$ with $X \neq Y$, we have $x \in \mathbb{N}_\lambda(a) \cap T^{-1}(X)$ and $y \in \mathbb{N}_\lambda(a) \cap T^{-1}(Y)$. By Proposition 6.3.7, there exists a sequence $x_o = x, x_1, \ldots, x_\ell$ such that for every j, $1 < j < \ell$, $x_j^{-1} = {}^*(x_{j-1}^{-1})$ in $\mathcal{D}_L(s_{t_j})$ with some $t_j \in \mathbb{Z}$, i.e. $x_j = x^*_{j-1}$ in $\mathcal{D}_R(s_{t_j})$, and $x_\ell^{-1} \in H_\lambda$. Clearly, the number ℓ is independent of the choice of a. Now we take $a,b \in \mathbb{Z}$ with $b > n$ and $a > b + 2\ell$. Then we have $x_j \in \mathbb{N}_\lambda(b)$ for any j, $0 < j < \ell$. Since $x \underset{L}{\sim} y$, we see by Theorem A(i) and C(ii) that there also exists a sequence $y_o = y, y_1, \ldots, y_\ell$ such that for every j, $1 < j < \ell$, $y_j = y^*_{j-1}$ in $\mathcal{D}_R(s_{t_j}) \cap \mathbb{N}_\lambda(b)$, $y_j \underset{L}{\sim} x_j$ and then $\mathcal{R}(y_j) = \mathcal{R}(x_j)$. In particular, $y_\ell \underset{L}{\sim} x_\ell$ and $\mathcal{R}(y_\ell) = \mathcal{R}(x_\ell)$. So by Proposition 11.3.3, we have $x_\ell, y_\ell \in T^{-1}(X^v)$ for some v, $0 < v < n$. Now we start with the pair $\{x_\ell, y_\ell\}$ and consider the sequence of pairs $\{x_\ell,y_\ell\}, \{x_{\ell-1},y_{\ell-1}\}, \ldots, \{x_o,y_o\} = \{x,y\}$. Then by repeatedly applying Lemma 11.4.2, we have $T(y_j) = T(x_j)$ for all j, $0 < j < \ell$. In particular, $T(y) = T(x)$. This gives a contradiction. So our conclusion is shown. □

<u>Proposition 11.4.4</u> <u>Any two elements x,y in the same left cell of A_n can be transformed from one to another by a succession of raising operations in $\mathbb{N}_\lambda$ and left star operations, where $\lambda = \sigma(x)$.</u>

<u>Proof</u> By Proposition 6.4.1, we have $y \in \sigma^{-1}(\lambda)$ by $x \underset{L}{\sim} y$ and $x \in \sigma^{-1}(\lambda)$. It follows from Propositions 6.3.7 and 7.4 that there exists $x', y' \in \bar{N}_\lambda$ such that $x' \underset{P_L}{\sim} x$ and $y' \underset{P_L}{\sim} y$. Proposition 11.4.3 tells us that $T(x') = T(y')$ since $x' \underset{L}{\sim} y'$. Let α be any number in the first column of $T(x')$. Then we see in the proof of Proposition 11.2.5 that both x' and y' can be transformed to the unique element of $\bar{N}_\lambda$ which satisfies $A(T(x'),\alpha)$ by a succession of raising operations in $\bar{N}_\lambda$ and left star operations. This implies that x and y can be transformed from one to another by a succession of raising operations in $\bar{N}_\lambda$ and left star operations. □

<u>Theorem 11.4.5</u> <u>For any $\lambda \in \Lambda_n$, $\sigma^{-1}(\lambda)$ is a disjoint union of $\dfrac{n!}{\prod_{j=1}^{m} \mu_j!}$ left (resp. right) cells of A_n, where $\mu = \{\mu_1 > \dots > \mu_m\}$ is the dual partition of λ.</u>

<u>Proof</u> By Propositions 10.2.5, 11.4.3 and Lemma 11.1.1, we see that $\sigma^{-1}(\lambda)$ is a disjoint union of $\dfrac{n!}{\prod_{j=1}^{m} \mu_j!}$ left cells of A_n. Since the map $w \to w^{-1}$ induces a bijection from the set of left cells of A_n to the set of right cells of A_n and since $\sigma^{-1}(\lambda)$ is invariant under this map by Lemma 3.3, the remaining result of this theorem follows. □

CHAPTER 12 : $\sigma^{-1}(\lambda)$ IS AN RL-EQUIVALENCE CLASS OF $\mathbf{A}_n$

Fix $\lambda = \{\lambda_1 > \dots > \lambda_r\} \in \Lambda_n$. We have shown in §6.4 that $\sigma^{-1}(\lambda)$ is a union of some RL-equivalence classes of $\mathbf{A}_n$. Now we shall prove that $\sigma^{-1}(\lambda)$ is an RL-equivalence class of $\mathbf{A}_n$. We may assume $\lambda \neq \{1 > \dots > 1\}$ and $n > 3$ by Chapter 4. We know from Proposition 6.3.7 that any element of $\sigma^{-1}(\lambda)$ is in the same left cell as some element of H_λ. But by Lemma 3.3, this is equivalent to saying that any element of $\sigma^{-1}(\lambda)$ is in the same right cell as some element of H_λ^{-1}, where $H_\lambda^{-1} = \{x \in \sigma^{-1}(\lambda) \mid x^{-1} \in H_\lambda\}$. By Proposition 11.3.4, we see that for any $x,y \in H_\lambda^{-1}$, if $\mathcal{R}(x) = \mathcal{R}(y)$ then $x \underset{L}{\sim} y$. So to reach our goal, it is sufficient to show that for any $x,y \in H_\lambda^{-1}$, there exists an element z of H_λ^{-1} such that $z \underset{R}{\sim} x$ and $\mathcal{R}(z) = \mathcal{R}(y)$. But this is equivalent to show that for any $x,y \in H_\lambda$, there exists an element z of H_λ such that $z \underset{L}{\sim} x$ and $\mathcal{L}(z) = \mathcal{L}(y)$. By Proposition 7.4 and the fact that $\mathbf{N}_\lambda \subseteq H_\lambda$, we need only show that for any $x \in \mathbf{N}_\lambda$, $y \in H_\lambda$, there exists an element z of $\mathbf{N}_\lambda$ such that $z \underset{L}{\sim} x$ and $\mathcal{L}(z) = \mathcal{L}(y)$. Note that for any $u,v \in H_\lambda$, $\mathcal{L}(u) = \mathcal{L}(v)$ if and only if there exists an integer i such that both u and v have a standard MDC form at i. Now assume that $x \in \mathbf{N}_\lambda$ has a standard MDC form at h and $y \in H_\lambda$ has a standard MDC form at k. We may assume $h < k < h + n$. Let $E = \{\overline{j_t^1(x) \mid 1} < t < r\} \subseteq \underline{n}$. Then by Proposition 10.2.3, we see that the element $z = (L_E)^k(x)$ has all required properties. Therefore, we get

<u>Theorem 12.1</u> <u>For any $n > 2$ and $\lambda \in \Lambda_n$, the set $\sigma^{-1}(\lambda)$ is an RL-equivalence class of $\mathbf{A}_n$.</u> □

CHAPTER 13 : LEFT CELLS ARE CHARACTERIZED BY THE GENERALIZED RIGHT τ-INVARIANT

In Chapter 11, we have characterized any left cell of A_n by a tabloid of size n by means of the map T which maps all the normalized elements of this left cell to the corresponding tabloid. In the present chapter, we shall give another characterization of any left cell of A_n, i.e. the generalized right τ-invariant which has been defined on an element of any Coxeter group in §1.1.

Let P_n be the standard parabolic subgroup of A_n generated by $\{s_1,\ldots,s_{n-1}\}$ which is isomorphic to the symmetric group S_n. In §13.2, we shall apply the above result to discuss the relations between the RL-equivalence classes (resp. left, right cells) in A_n and in P_n. Our main results in §13.2 are as follows: The intersection of any RL-equivalence class of A_n with P_n is non-empty and is just an RL-equivalence class of P_n. The intersection of any left (resp. right) cell of A_n with P_n is either empty or a left (resp. right) cell of P_n.

In the proof of these results, we actually give a proof for determination of all left, right cells and all RL-equivalence classes of the symmetric group S_n in our own way which differs from Lusztig [1] and Vogan [8].

§13.1 LEFT CELLS ARE CHARACTERIZED BY THE GENERALIZED RIGHT τ-INVARIANT

<u>Lemma 13.1.1</u> <u>If $w,y \in A_n$ have the same generalized right τ-invariant, then $w \underset{RL}{\sim} y$.</u>

<u>Proof</u> By Theorem 12.1, it is sufficient to show that $\sigma(w) = \sigma(y)$. Assume $\lambda = \sigma(w)$ and $\lambda' = \sigma(y)$. Then $w^{-1} \in \sigma^{-1}(\lambda)$ and $y^{-1} \in \sigma^{-1}(\lambda')$ by Lemma 3.3. Hence by Proposition 6.3.7, there exists a sequence $w_o = w^{-1}, w_1, \ldots, w_\ell$ such that for every j, $1 < j < \ell$, $w_j = {}^*w_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ for some $s_{i_j} \in \Delta$ and $w_\ell \in H_\lambda$. Since y has the same generalized right τ-invariant as w, this implies that there also exists a sequence $y_o = y^{-1}, y_1, \ldots, y_\ell$ such that for every j, t with $1 < j < \ell$ and $0 < t < \ell$, $y_j = {}^*y_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ and $\mathcal{L}(y_t) = \mathcal{L}(w_t)$. In particular, $\mathcal{L}(y_\ell) = \mathcal{L}(w_\ell)$. So

$\lambda' = \sigma(y) = \sigma(y^{-1}) = \sigma(y_\ell) > \pi(\mathcal{L}(y_\ell)) = \pi(\mathcal{L}(w_\ell)) = \sigma(w_\ell) = \sigma(w^{-1})$

$= \sigma(w) = \lambda$. i.e. $\lambda' > \lambda$. By symmetry, we also have $\lambda > \lambda'$. So $\lambda = \lambda'$. □

<u>Theorem 13.1.2</u> <u>For $w, y \in A_n$, $w \underset{L}{\sim} y \Leftrightarrow w, y$ have the same generalized right τ-invariant.</u>

<u>Proof</u> ($\Rightarrow$) By Theorems A and C.

($\Leftarrow$) By Lemma 13.1.1, we have $w, y \in \sigma^{-1}(\lambda)$ for some $\lambda \in \Lambda_n$. Then $w^{-1}, y^{-1} \in \sigma^{-1}(\lambda)$. Since w^{-1}, y^{-1} have the same generalized left τ-invariant, there exist two sequences $w_o = w^{-1}, w_1, \ldots, w_\ell$ and $y_o = y^{-1}, y_1, \ldots, y_\ell$ such that for every j, $1 < j < \ell$, $w_j = {}^*w_{j-1}$ and $y_j = {}^*y_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ with some $s_{i_j} \in \Delta$, and $\mathcal{L}(w_h) = \mathcal{L}(y_h)$, $0 < h < \ell$, $w_\ell, y_\ell \in H_\lambda$. By Proposition 11.3.4, we have $w_\ell^{-1} \underset{L}{\sim} y_\ell^{-1}$, i.e. $w_\ell \underset{R}{\sim} y_\ell$. Owing to Theorem C, we get $w_j \underset{R}{\sim} y_j$ for all j. In particular, $w^{-1} \underset{R}{\sim} y^{-1}$. So $w \underset{L}{\sim} y$. □

§13.2 THE STANDARD PARABOLIC SUBGROUP $\mathcal{P}_n$

For any $w \in \mathcal{A}_n$, let $\{e_w(u,j_u) \mid u \in \mathbb{Z}\}$ be the entry set of w. Then we see that $w \in \mathcal{P}_n$ if and only if j_u satisfies $1 \leq j_u \leq n$ for any u, $1 \leq u \leq n$.

Now assume that w is in $\mathcal{P}_n$. By the property that $j_{u+n} = j_u+n$ for any $u \in \mathbb{Z}$, we see that if $u,v \in \mathbb{Z}$ satisfy $u-v \geq n$ then $j_u > j_v$. So for any descending chain $\{e_w(u_i,j_{u_i}) \mid 1 \leq i \leq t,\ u_1 < u_2 < \ldots < u_t\}$ of entries of w, there must exist some $q \in \mathbb{Z}$ such that $qn + 1 \leq u_1 < u_2 < \ldots < u_t \leq (q+1)n$.

Bearing these facts in mind, we shall first show that any element x of $\mathcal{P}_n$ can be transformed to some element of $\mathcal{P}_n \cap H_\lambda$ by a succession of left star operations in $\mathcal{P}_n$, where $\lambda = \sigma(x)$.

<u>Lemma 13.2.1</u> <u>For $w \in \mathcal{A}_n$, let $A(w)$ be the block of w consisting of rows from the $(i+1)$-th to the $(i+j)$-th with $1 \leq j \leq n$. Let $S = \{e_w(u_\alpha,j_{u_\alpha}) \mid {}^{1 \leq \alpha \leq t}_{u_1<u_2<\ldots<u_t}\}$ be a longest descending chain of w in $A(w)$. Then we can transform w to w' by a succession of left star operations on the block $A(w)$ such that w' has a descending chain $\{e_{w'}(u,j'_u) \mid i+j+1-t \leq u \leq i+j\}$.</u>

<u>Remark:</u> We say that a left star operation on w acts on the block $A(w)$ if it is defined in $\mathcal{D}_L(s_u)$ with $i+1 \leq u \leq i+j-2$, where i,j are defined as above.

<u>Proof</u> Clearly, t satisfies $1 \leq t \leq j$. Fix $t \geq 1$, let us apply induction on $j-t \geq 0$. It is trivial for the case $j-t = 0$, since in that case, we just take $w' = w$. Now assume $j-t > 0$.

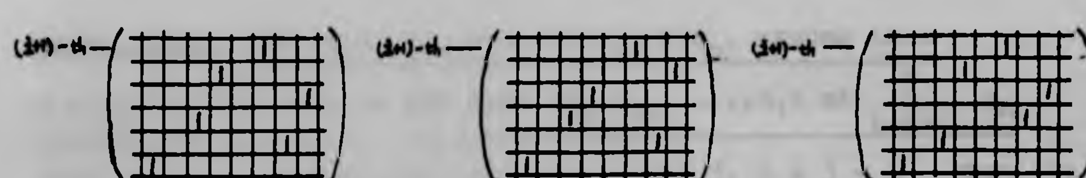
(i) If $u_1 \neq i+1$, let $A'(w)$ be the block of w consisting of rows from the $(i+2)$-th to the $(i+j)$-th. Then S is a longest descending chain of w in $A'(w)$ and so our result follows from the inductive hypothesis.

(ii) If $u_1 = i+1$ and $j_{i+1} < j_{i+2}$, where $e_w(i+2, j_{i+2})$ is an entry of w, then $e_w(i+2, j_{i+2})$ is not in S. Let $S' = S \cup \{e_w(i+2, j_{i+2})\} - \{e_w(i+1, j_{i+1})\}$. Then S' is also a longest descending chain of w in $A(w)$. This reduces to case (i).

(iii) Let $u_1 = i+1$ and $j_{i+1} > j_{i+2} > \dots > j_{i+h}$ and $j_{i+h} < j_{i+h+1}$ with $h > 2$, where $e_w(i+k, j_{i+k})$ are the entries of w for all k, $1 < k < h+1$. By our assumption, we see that h is less than j. Then there exists $x \in A_n$ with $w \xrightarrow{*(i+1,h,1)} x$. Clearly, x is obtained from w by a succession of left star operations in the block $A(w)$ and the maximal length of the descending chains of x in $A(x)$ is also equal to t by Lemma 3.7 and its proof, where $A(x)$ is the block of x consisting of rows from the $(i+1)$-th to the $(i+j)$-th. Let $e_x(i+1, j'_{i+1})$ and $e_x(i+2, j'_{i+2})$ be two entries of x. Then we have $j'_{i+1} < j'_{i+2}$. This reduces to case (i

Therefore our conclusion follows by induction. □

The following is an example for Lemma 13.2.1, where w, x, $w' \in A_n$ with $n > 6$, $j = 6$, $t = 4$, $w \xrightarrow{*(i+1,2,1)} x$ and $x \xrightarrow{*(i+2,3,1)} w'$



w x w'

Lemma 13.2.2 For $w \in A_n$, let A be the block of w consisting of rows from the (i+1)-th to the (i+j)-th with $1 \leq j \leq n$. Assume that $D = \{e_w(u,j_u) \mid i+j+1-t \leq u \leq i+j\}$ is a longest descending chain of w in A. Assume that $S = S_1 \cup \ldots \cup S_h$ is any disjoint union of h descending chains of w in A. Then there exists a disjoint union $S' = S_1' \cup \ldots \cup S_h'$ of h descending chains of w in A such that $|S'| \geq |S|$ and $S_1' = D$.

Proof We may assume $S \cap D \neq \emptyset$, as otherwise, $S' = D \cup S_2 \cup \ldots \cup S_h$ would be as required.

Let k be the smallest integer with $i+j+1-t \leq k \leq i+j$ such that $e_w(k,j_k) \in S$. We may assume $e_w(k,j_k) \in S_1$. Let $S' = S_1' \cup \ldots \cup S_h'$ be as follows:

$$S_1' = S_1 \cup \{e_w(u,j_u) \mid k \leq u \leq i+j\}$$

$$S_\ell' = S_\ell - \{e_w(u,j_u) \mid k \leq u \leq i+j\}, \qquad 2 \leq \ell \leq h.$$

Then $S' = S_1' \cup \ldots \cup S_h'$ is a disjoint union of h descending chains of w in A satisfying $|S'| \geq |S|$. If $k = i+j+1-t$, then S' is as required. If not, then $S'' = D \cup S_2' \cup \ldots \cup S_h'$ is as required since $|D| \geq |S_1'|$. □

Lemma 13.2.3 For $\lambda = \{\lambda_1 \geq \ldots \geq \lambda_r\} \in \Lambda_n$, assume that $w \in \mathcal{P}_n \cap \sigma^{-1}(\lambda)$ has an MDC form $(A_k, A_{k-1}, \ldots, A_1)$ at $\sum_{h=k+1}^{r} \lambda_h$ with $0 \leq k \leq r$ and $|A_j(w)| = \lambda_j$ for every j, $1 \leq j \leq k$. Then the maximal length ℓ of the descending chains of w in B is λ_{k+1}, where

B is the block of w consisting of rows from the 1-st to the $(\sum_{h=k+1}^{r} \lambda_h)$-th.

<u>Proof</u> By our assumption, it is easily seen that $\ell < \lambda_{k+1}$. Let B_j be the block of w consisting of rows from the 1-st to the $(\sum_{h=j}^{r} \lambda_h)$-th for any j, $1 < j < k+1$. Then $B_{k+1} = B$. By the remark at the beginning of this section, we see that for any h with $1 < h < r$, there exists a disjoint union $S = S_1 \cup \ldots \cup S_h$ of h descending chains of w in B_1 satisfying $C_n(w,h)$ and $|S| = \sum_{j=1}^{h} \lambda_j$. Now we take $h = k+1$. Since A_1 is a longest descending chain of w in B_1, it follows from Lemma 13.2.2 that there exists a disjoint union $S^{(1)} = S_1^{(1)} \cup \ldots \cup S_{k+1}^{(1)}$ of k+1 descending chains of w in B_1 satisfying $C_n(w,k+1)$, $|S^{(1)}| > |S|$ and $S_1^{(1)} = A_1$. Since $w \in \sigma^{-1}(\lambda)$, we must have $|S^{(1)}| = |S|$. Now $\tilde{S}^{(1)} = S_2^{(1)} \cup \ldots \cup S_{k+1}^{(1)}$ is a disjoint union of k descending chains of w in B_2.satisfying $C_n(w,k)$ and $|\tilde{S}^{(1)}| = \sum_{h=2}^{k+1} \lambda_h$.

Since A_2 is a longest descending chain of w in B_2, it follows from Lemma 13.2.2 that there exists a disjoint union $S^{(2)} = S_2^{(2)} \cup \ldots \cup S_{k+1}^{(2)}$ of k descending chains of w in B_2 satisfying $C_n(w,k)$, $|S^{(2)}| > |\tilde{S}^{(1)}|$ and $S_2^{(2)} = A_2$. Again by the fact $w \in \sigma^{-1}(\lambda)$, we see $|S^{(2)}| = |\tilde{S}^{(1)}| = \sum_{h=2}^{k+1} \lambda_h$. In such a way, for any j with $1 < j < k$, we see that there exists a disjoint union $S^{(j)} = S_j^{(j)} \cup \ldots \cup S_{k+1}^{(j)}$ of k+2-j descending chains of w in B_j satisfying $C_n(w,k+2-j)$, $|S^{(j)}| = \sum_{h=j}^{k+1} \lambda_h$ and $S_j^{(j)} = A_j$.

In particular, when $j = k$, we have a disjoint union $S^{(k)} = S_k^{(k)} \cup S_{k+1}^{(k)}$ of 2 descending chains of w in B_k satisfying $C_n(w,2)$, $S_k^{(k)} = A_k$ and $|S^{(k)}| = \sum_{h=k}^{k+1} \lambda_h$. So $S_{k+1}^{(k)}$ is a descending chain of w in B with $|S_{k+1}^{(k)}| = \lambda_{k+1}$. This implies $\ell = \lambda_{k+1}$. □

<u>Proposition 13.2.4</u> <u>For any $\lambda = \{\lambda_1 > \ldots > \lambda_r\} \in \Lambda_n$, if $w \in \sigma^{-1}(\lambda) \cap \mathcal{P}_n$, then there exists a sequence of elements $w_o = w, w_1, \ldots, w_t$ in $\mathcal{P}_n$ such that for every j, $1 < j < t$, $w_j = {}^*w_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ with some i_j, $1 < i_j < n-2$ and $w_t \in H_\lambda \cap \mathcal{P}_n$.</u>

<u>Proof</u> Let B_j, $1 < j < r$, be defined as in the proof of Lemma 13.2.3. By Lemma 13.2.3, if w has an MDC form $(A_k, \ldots, A_1)$ at $\sum_{h=k+1}^{r} \lambda_h$ with $0 < k < r$ and $|A_j(w)| = \lambda_j$ for every j, $1 < j < k$, then the maximal length of the descending chains of w in B_{k+1} is λ_{k+1}. So by Lemma 13.2.1, we can transform w to x by a succession of left star operations in the block B_{k+1} such that x has an MDC form $(A_{k+1}, \ldots, A_1)$ at $\sum_{h=k+2}^{r} \lambda_h$ with $|A_j(x)| = \lambda_j$ for every j, $1 < j < k+1$. Now we go through the same process by taking $k = 0, 1, \ldots, r-1$ in turn. Finally we get an element w' of $\mathcal{P}_n$ which has an MDC form $(A_r, \ldots, A_1)$ at 0 with $|A_j(w')| = \lambda_j$ for every j, $1 < j < r$, i.e. $w' \in H_\lambda$. Since the whole process from w to w' is a succession of left star operations in $\mathcal{P}_n$, this implies our result. □

<u>Corollary 13.2.5</u> <u>For any $\lambda \in \Lambda_n$, if $w \in \sigma^{-1}(\lambda) \cap \mathcal{P}_n$, then there</u>

exists a sequence of elements $w_o = w, w_1, \ldots, w_t$ in $\mathcal{P}_n$ such that for every j, $1 \leq j \leq t$, either $w_j = {}^*w_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ or $w_j = w_{j-1}^*$ in $\mathcal{D}_R(s_{i_j})$ with some i_j, $1 \leq i_j \leq n-2$, and w_t, w_t^{-1} are both in $\mathcal{P}_n \cap H_\lambda$.

Proof By Proposition 13.2.4, we may assume $w \in \mathcal{P}_n \cap H_\lambda$. Clearly w^{-1} is also in the set $\sigma^{-1}(\lambda) \cap \mathcal{P}_n$. Also by Proposition 13.2.4, there exists a sequence of elements $w_o = w^{-1}, w_1, \ldots, w_t$ in $\mathcal{P}_n$ such that for every j, $1 \leq j \leq t$, $w_j = {}^*w_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ with some i_j, $1 \leq i_j \leq n-2$, and $w_t \in H_\lambda$. i.e. there exists a sequence of elements $w_o^{-1} = w, w_1^{-1}, \ldots, w_t^{-1}$ in $\mathcal{P}_n$ such that $w_j^{-1} = (w_{j-1}^{-1})^*$ in $\mathcal{D}_R(s_{i_j})$ for every j, $1 \leq j \leq t$ and $w_t \in H_\lambda$. Since this sequence is in $\sigma^{-1}(\lambda)$ and also in some left cell of $\mathcal{P}_n$, we have $\mathcal{L}(w_t^{-1}) = \mathcal{L}(w_{t-1}^{-1}) = \ldots = \mathcal{L}(w)$ by Theorem A and hence $w_t^{-1} \in H_\lambda$. Our assertion follows. □

Now we can consider the relations between the RL-equivalence classes (resp. left, right cells) in A_n and in $\mathcal{P}_n$.

For any $\lambda = \{\lambda_1 \geq \ldots \geq \lambda_r\} \in \Lambda_n$, let $J_\lambda(0) = J_r \cup \ldots \cup J_1 \subset \Delta$ with $J_h = \{s_{\alpha_h+1}, s_{\alpha_h+2}, \ldots, s_{\alpha_h+\lambda_h-1}\}$, $\alpha_h = \sum_{\ell=h+1}^{r} \lambda_\ell$. Then it is clear that $w_o^{J_\lambda(0)} \in \mathcal{P}_n \cap \sigma^{-1}(\lambda)$. So we get the following result.

Proposition 13.2.6 For any $\lambda \in \Lambda_n$, the intersection of any RL-equivalence class of A_n with $\mathcal{P}_n$ is non-empty.

Proof By Theorem 12.1, any RL-equivalence class of A_n has the form $\sigma^{-1}(\lambda)$ for some $\lambda \in \Lambda_n$. Thus our result follows immediately. $\square$

By Proposition 13.2.6, we see that the intersection of any RL-equivalence class of $\mathcal{A}_n$ with $\mathcal{P}_n$ is non-empty and hence it is a union of some RL-equivalence classes of $\mathcal{P}_n$. Now we shall show that such an intersection is actually an RL-equivalence class of $\mathcal{P}_n$.

<u>Lemma 13.2.7</u> <u>For $\lambda = \{\lambda_1 > \ldots > \lambda_r\} \in \Lambda_n$, assume that $w, w^{-1} \in H_\lambda \cap \mathcal{P}_n$. Then $w = w_o^{J_\lambda(0)}$.</u>

<u>Proof</u> By the assumption that $w \in H_\lambda \cap \mathcal{P}_n$, w has a standard MDC form $(A_r, \ldots, A_1)$ at 0. Let $e(i_t^u(w), j_t^u(w))$ be the u-th entry of $A_t(w)$. Then $i_t^u(w) = \sum_{h=t+1}^{r} \lambda_h + u$. We claim $j_1^1(w) = n$. For suppose that $j_1^1(w) < n$. Then there exists an entry $e_w(i,n)$ of w with $1 \le i \le n$ and $i \neq i_1^1(w)$. We have $i \not> i_1^1(w)$ by the fact that $A_1(w)$ is a DC block. So we must have $i < i_1^1(w)$. But then $S = \{e_w(i,n)\} \cup A_1$ is a descending chain of w with $|S| = \lambda_1+1$. This contradicts $w \in \sigma^{-1}(\lambda)$. So w has the entry $e_w(n+1-\lambda_1, n)$. Also by the assumption that $w^{-1} \in H_\lambda \cap \mathcal{P}_n$ and the fact that the matrix w^{-1} is the transpose of w, we see that $e_{w^{-1}}(n+1-\lambda_1, n)$ is an entry of w^{-1} and so $e_w(n, n+1-\lambda_1)$ is an entry of w, i.e.

. Now $n = j_1^1(w) > j_1^2(w) > \ldots > j_1^{\lambda_1}(w) = n+1-\lambda_1$.

So we get $j_1^h(w) = n+1-h$ for any h, $1 \le h \le \lambda_1$.

Now suppose that we have shown $j_t^u(w) = \sum_{h=t}^{r} \lambda_h + 1 - u$ for some k, $1 \le k < r$ and all t, u with $1 \le t \le k$ and $1 \le u \le \lambda_t$.

Then we have $1 \leq j^{u'}_{h'}(w) \leq \sum_{h=k+1}^{r} \lambda_h$ for any h', u' with $k < h' \leq r$ and $1 \leq u' \leq \lambda_{h'}$. Note that the maximal length of the descending chains of w in its submatrix D is λ_{k+1}, where D is the $(\sum_{h=k+1}^{r} \lambda_h) \times (\sum_{h=k+1}^{r} \lambda_h)$ matrix lying between the 1st row (resp. column) and the $(\sum_{h=k+1}^{r} \lambda_h)$-th row (resp. column) of w. By the same argument as above with the number $\sum_{h=k+1}^{r} \lambda_h$ in place of n, we can show that $j^{u}_{k+1}(w) = \sum_{h=k+1}^{r} \lambda_h + 1 - u$ for any u, $1 \leq u \leq \lambda_{k+1}$. So by applying induction on $k \geq 1$, finally we see that $j^{u}_{t}(w) = \sum_{h=t}^{r} \lambda_h + 1-u$ for any t, u with $1 \leq t \leq r$ and $1 \leq u \leq \lambda_t$. This implies $w = w^{J_\lambda(0)}$. □

The following result gives a one-to-one correspondence between the set of RL-equivalence classes of A_n and those of P_n.

<u>Theorem 13.2.8 For any $\lambda \in \Lambda_n$, the set $P_n \cap \sigma^{-1}(\lambda)$ is an RL-equivalence class of P_n. Conversely, any RL-equivalence class of P_n has such a form.</u>

<u>Proof</u> By Theorem 12.1 and Proposition 13.2.6, it is sufficient to show that any two elements of $P_n \cap \sigma^{-1}(\lambda)$ are in the same RL-equivalence class of P_n. But by Corollary 13.2.5 and Lemma 13.2.7, they are all P-equivalent to $w_o^{J_\lambda(0)}$. So our result follows by Theorem D(iii). □

Now we consider the relations between the left (resp. right) cells in $\mathcal{A}_n$ and in $\mathcal{P}_n$.

In general, the intersection of any left (resp. right) cell of $\mathcal{A}_n$ with $\mathcal{P}_n$ is not necessarily non-empty. But what will happen when such an intersection is non-empty? Clearly we can at least say that it is a union of some left (resp. right) cells of $\mathcal{P}_n$. The following result gives us a more satisfactory answer.

<u>Theorem 13.2.9</u> <u>The intersection of any left (resp. right) cell of $\mathcal{A}_n$ with $\mathcal{P}_n$ is either empty or a left (resp. right) cell of $\mathcal{P}_n$. Conversely, any left (resp. right) cell of $\mathcal{P}_n$ can be expressed as an intersection of some left (resp. right) cell of $\mathcal{A}_n$ with $\mathcal{P}_n$.</u>

<u>Proof</u> Let $x, y \in \mathcal{P}_n$. It is sufficient to show that if x and y are in the same left cell of $\mathcal{A}_n$ then they are in the same left cell of $\mathcal{P}_n$.

Now assume that x and y are in the same left cell of $\mathcal{A}_n$. By Proposition 6.4, we have $\sigma(x) = \sigma(y) = \lambda$ for some $\lambda \in \Lambda_n$. By Proposition 13.2.4, we may assume $x, y \in H_\lambda \cap \mathcal{P}_n$. Lemma 3.3 and Theorem 13.1.2 tell us that x^{-1}, y^{-1} are in $\sigma^{-1}(\lambda)$ and, they have the same generalized left τ-invariant in $\mathcal{A}_n$ and hence in $\mathcal{P}_n$. So by Proposition 13.2.4, there exist two sequences of elements $x_o = x^{-1}, x_1, \ldots, x_t$ and $y_o = y^{-1}, y_1, \ldots, y_t$ in $\mathcal{P}_n$ such that for every j, k with $1 < j < t$ and $0 < k < t$, $x_j = {}^*x_{j-1}$ and $y_j = {}^*y_{j-1}$ in $\mathcal{D}_L(s_{i_j})$ with some i_j, $1 < i_j < n-2$, $\mathcal{L}(x_k) = \mathcal{L}(y_k)$ and $x_t, y_t \in H_\lambda \cap \mathcal{P}_n$. Since $x_o = x^{-1}, x_1, \ldots, x_t$ are all in the same left cell of $\mathcal{P}_n$, we have $\mathcal{R}(x_t) = \mathcal{R}(x^{-1}) = \mathcal{L}(x)$ and thus $x_t^{-1} \in H_\lambda \cap \mathcal{P}_n$. By Lemma 13.2.7, we must have $x_t = w_o^{J_\lambda(0)}$. Similarly we can show $y_t = w_o^{J_\lambda(0)}$ and so $x_t = y_t$. Thus $x_j = y_j$ for all j. In particular, $x^{-1} = y^{-1}$, i.e. $x = y$. So x and y are in the same left cell of $\mathcal{P}_n$. Our proof is complete. □

We know that any standard parabolic subgroup of A_n which is isomorphic to the symmetric group S_n has the form $\mathcal{P}_t = \langle s_i | 1 \le i \le n, i \ne t\rangle$ for some t, $1 \le t \le n$. Clearly, the automorphism ϕ^t of A_n fixes all $\sigma^{-1}(\lambda)$, $\lambda \in \Lambda_n$ and induces an isomorphism from $\mathcal{P}_n$ to $\mathcal{P}_t$ which preserves the left, right and two-sided cells as well as RL-equivalence classes for them. So, if we replace $\mathcal{P}_n$ by $\mathcal{P}$, where $\mathcal{P} \in \{\mathcal{P}_t | 1 \le t \le n\}$, then all results in this section still hold.

CHAPTER 14 - THE TWO-SIDED CELLS OF THE AFFINE WEYL GROUP A_n

We have shown that for any $\lambda \in \Lambda_n$, $\sigma^{-1}(\lambda)$ is a RL-equivalence class of A_n which consists of $\frac{n!}{\sum_{j=1}^{m} \mu_j!}$ left (resp. right) cells of A_n, where $\mu = \{\mu_1 > \dots > \mu_m\}$ is the dual partition of λ. From this, we know that the total number of left (resp. right) cells of A_n is

$$\sum_{\{\lambda_1 > \dots > \lambda_r\} \in \Lambda_n} \frac{n!}{\prod_{j=1}^{r} \lambda_j!}$$

Since any two-sided cell of A_n is a union of some RL-equivalence classes of A_n, this implies that the number of two-sided cells of A_n is finite and is less than or equal to the number of partitions of n. So far, all the results here have been obtained by our own elementary methods and do not rely on the knowledge of the intersection cohomology theory. But now we shall use a recent result of Lusztig, i.e. Theorem E [6] in Chapter 1 to prove that any RL-equivalence class of A_n is actually a two-sided cell of A_n. Then we can determine the exact number of two-sided cells of A_n which is equal to the number of partitions of n.

This result of Lusztig comes from the deep theory of intersection cohomology.

To show our result, we shall first give some lemmas. The notation $y \underset{\Gamma}{\sim} w$ means that y, w are in the same two-sided cell of A_n.

Lemma 14.1 If $y, w \in A_n$ with $y \underset{\Gamma}{\sim} w$, then

(i) $y \underset{L}{\leq} w$ implies $\mathcal{R}(y) = \mathcal{R}(w)$.

(ii) $y \underset{R}{\leq} w$ implies $\mathcal{L}(y) = \mathcal{L}(w)$.

Proof It is enough to show (i). Since $y \underset{L}{\leq} w$, there exists a sequence $y_o = y, y_1, \ldots, y_t = w$ such that either $y_{i-1} \prec y_i$ or $y_i \prec y_{i-1}$, and $\mathcal{L}(y_{i-1}) \not\subset \mathcal{L}(y_i)$ for every i, $1 < i < t$. Clearly, $y = y_o \underset{L}{\leq} y_1 \underset{L}{\leq} \ldots \underset{L}{\leq} y_t = w$. Since $y \underset{\Gamma}{\sim} w$, this implies that all y_i, $0 < i < t$, are in the same two-sided cell of A_n. By Theorem A(i), we have

$$\mathcal{R}(y) = \mathcal{R}(y_o) \supseteq \mathcal{R}(y_1) \supseteq \ldots \supseteq \mathcal{R}(y_t) = \mathcal{R}(w).$$

In particular, $\mathcal{R}(y) \supseteq \mathcal{R}(w)$. If $\mathcal{R}(y) \supsetneq \mathcal{R}(w)$, then there exists at least one j, $1 < j < r$, such that $\mathcal{R}(y_{j-1}) \supsetneq \mathcal{R}(y_j)$, i.e. $\mathcal{R}(y_{j-1}) \not\subset \mathcal{R}(y_j)$. So by Theorem E, y_{j-1} and y_j are not in the same two-sided cell. This gives a contradiction. Hence we must have $\mathcal{R}(y) = \mathcal{R}(w)$. □

Lemma 14.2 Let $y, w \in A_n$ with $y \underset{\Gamma}{\sim} w$.

(i) Let y, w be two elements in $D_L(s_t)$. If $y \underset{R}{\leq} w$, then ${}^*y \underset{R}{\leq} {}^*w$ and ${}^*y \underset{\Gamma}{\sim} {}^*w$.

(ii) <u>Let y,w be two elements in $\mathcal{D}_R(s_t)$. If $y \underset{L}{\leq} w$, then</u> <u>$y^* \underset{L}{\leq} w^*$ and $y^* \underset{\Gamma}{\sim} w^*$.</u>

<u>Proof</u> We first note that, if $x \in \mathcal{D}_L(s_u)$, then $*x \underset{L}{\sim} x$, hence, by Theorem A(i), we have $\mathcal{R}(*x) = \mathcal{R}(x)$. Now let y, w be two elements in $\mathcal{D}_L(s_t)$ such that $y \underset{R}{\leq} w$ and $y \underset{\Gamma}{\sim} w$. Then there exists a sequence $y_0 = y, y_1, \ldots, y_r = w$ such that either $y_{i-1} \prec y_i$ or $y_{i-1} \succ y_i$, and $\mathcal{R}(y_{i-1}) \not\subset \mathcal{R}(y_i)$ for every i, $1 \leq i \leq r$. Clearly,

$$y = y_0 \underset{R}{\leq} y_1 \underset{R}{\leq} \cdots \underset{R}{\leq} y_r = w.$$

Since $y \underset{\Gamma}{\sim} w$, this implies that all y_i, $0 \leq i \leq r$, are in the same two-sided cell. So by Lemma 14.1(ii), we have

$$\mathcal{L}(y) = \mathcal{L}(y_0) = \mathcal{L}(y_1) = \cdots = \mathcal{L}(y_r) = \mathcal{L}(w).$$

Hence, $*y_i$, $0 \leq i \leq r$, are well defined. Theorem B shows that either $*y_{i-1} \prec *y_i$ or $*y_i \prec *y_{i-1}$ for every i, $1 \leq i \leq r$. By the remark at the beginning of the proof, we have $\mathcal{R}(y_i) = \mathcal{R}(*y_i)$ for all i. It follows that $\mathcal{R}(*y_{i-1}) \not\subset \mathcal{R}(*y_i)$ for i, $1 \leq i \leq r$ and hence $*y = *y_0 \underset{R}{\leq} *y_1 \underset{R}{\leq} \cdots \underset{R}{\leq} *y_r = *w$. In particular, $*y \underset{R}{\leq} *w$. On the other hand, we have $*y \underset{L}{\sim} y \underset{\Gamma}{\sim} w \underset{L}{\sim} *w$ and hence $*y \underset{\Gamma}{\sim} *w$. (i) is proved. The proof of (ii) is entirely similar. □

<u>Lemma 14.3</u> <u>Let y, w $\in A_n$ with $y \underset{\Gamma}{\sim} w$. Then</u>

(i) <u>$y \underset{L}{\leq} w$ implies $y \underset{L}{\sim} w$.</u>

(ii) <u>$y \underset{R}{\leq} w$ implies $y \underset{R}{\sim} w$.</u>

<u>Proof</u> It is enough to show (i). Assume $y \in \sigma^{-1}(\lambda)$ and $w \in \sigma^{-1}(\lambda')$ for some $\lambda, \lambda' \in \Lambda_n$. Then $y^{-1} \in \sigma^{-1}(\lambda)$ by Lemma 3.3. So by Proposition 6.3.7, there exists a sequence $y_o = y^{-1}, y_1, \ldots, y_r$ in $\sigma^{-1}(\lambda)$ such that for every j, $1 \leq j \leq r$, $y_j = {}^*y_{j-1}$ in $\mathcal{D}_L(s_{\alpha_j})$ with some $s_{\alpha_j} \in \Delta$, and $y_r \in H_\lambda$. Now we define a sequence $w_o, w_1, \ldots, w_r$ as follows: Let $w_o = w^{-1}$. Then by $y \underset{\Gamma}{\sim} w$ and $y \underset{L}{\leq} w$, we have $y_o \underset{\Gamma}{\sim} w_o$ and $y_o \underset{R}{\leq} w_o$. Thus by Lemma 14.1(ii), we have $\mathcal{L}(y_o) = \mathcal{L}(w_o)$ and then $w_o \in \mathcal{D}_L(s_{\alpha_1})$. Let $w_1 = {}^*w_o$ in $\mathcal{D}_L(s_{\alpha_1})$. By Lemma 14.2(i), we see that $y_1 \underset{R}{\leq} w_1$ and $y_1 \underset{\Gamma}{\sim} w_1$. Again by Lemma 14.1(ii), we have $\mathcal{L}(y_1) = \mathcal{L}(w_1)$ and then $w_1 \in \mathcal{D}_L(s_{\alpha_2})$. Let $w_2 = {}^*w_1$ in $\mathcal{D}_L(s_{\alpha_2})$. In such a way, we get a sequence $w_o = w^{-1}, w_1, \ldots, w_r$ such that $w_j = {}^*w_{j-1}$ in $\mathcal{D}_L(s_{\alpha_j})$, $y_h \underset{R}{\leq} w_h$, $y_h \underset{\Gamma}{\sim} w_h$ and $\mathcal{L}(y_h) = \mathcal{L}(w_h)$ for every j, h with $1 \leq j \leq r$ and $0 \leq h \leq r$. In particular, $\mathcal{L}(w_r) = \mathcal{L}(y_r)$. So

$$\lambda' = \sigma(w) = \sigma(w^{-1}) = \sigma(w_r) \geq \pi(\mathcal{L}(w_r)) = \pi(\mathcal{L}(y_r)) = \sigma(y_r) = \lambda.$$

On the other hand, by Lemma 3.3, $w^{-1} \in \sigma^{-1}(\lambda')$. So by Proposition 6.3.7, there exists a sequence $x_o = w^{-1}, x_1, \ldots, x_t$ in $\sigma^{-1}(\lambda')$ such that for every j, $1 \leq j \leq t$, $x_j = {}^*x_{j-1}$ in $\mathcal{D}_L(s_{\beta_j})$ with some $s_{\beta_j} \in \Delta$ and $x_t \in H_{\lambda'}$. Then we can define a sequence $z_o = y^{-1}, z_1, \ldots, z_t$ from the sequence $x_o, x_1, \ldots, x_t$ in the similar way as $w_o, w_1, \ldots, w_r$ from $y_o, y_1, \ldots, y_r$ such that $z_j = {}^*z_{j-1}$ in $\mathcal{D}_L(s_{\beta_j})$, $z_h \underset{R}{\leq} x_h$, $z_h \underset{\Gamma}{\sim} x_h$ and $\mathcal{L}(z_h) = \mathcal{L}(x_h)$ for every j, h with $1 < j < t$ and $0 < h < t$. In particular, $\mathcal{L}(z_t) = \mathcal{L}(x_t)$. So

$$\lambda = \sigma(y) = \sigma(y^{-1}) = \sigma(z_t) > \pi(\mathcal{L}(z_t)) = \pi(\mathcal{L}(x_t)) = \sigma(x_t) = \lambda'.$$

This implies $\lambda' = \lambda$ and then y_r, $w_r \in H_\lambda$ with $\mathcal{L}(y_r) = \mathcal{L}(w_r)$. So $(y_r^{-1})^{-1}, (w_r^{-1})^{-1} \in H_\lambda$ with $\mathcal{R}(y_r^{-1}) = \mathcal{R}(w_r^{-1})$. By Proposition 11.3.4, we have $y_r^{-1} \underset{L}{\sim} w_r^{-1}$. Therefore, by repeatedly applying Theorem C(ii) to the sequences $y_o^{-1} = y, y_1^{-1}, \ldots, y_r^{-1}$ and $w_o^{-1} = w, w_1^{-1}, \ldots, w_r^{-1}$, we get $y_j^{-1} \underset{L}{\sim} w_j^{-1}$ for all j, $0 < j < r$. In particular, $y \underset{L}{\sim} w$. □

We now can determine all the two-sided cells of $\mathbb{A}_n$.

<u>Theorem 14.4</u> <u>The set $\sigma^{-1}(\lambda)$ is a two-sided cell of $\mathbb{A}_n$ for any $\lambda \in \Lambda_n$. Conversely, any two-sided cell of $\mathbb{A}_n$ has such a form.</u>

<u>Proof</u> We have already shown that $\sigma^{-1}(\lambda)$ is in some two-sided cell of $\mathbb{A}_n$. So it is sufficient to show that for any y, $w \in \mathbb{A}_n$ with $y \underset{\Gamma}{\sim} w$, we have $\sigma(y) = \sigma(w)$. Now assume $y \underset{\Gamma}{\sim} w$. Then there exists a sequence $y_o = y, y_1, \ldots, y_r = w$ in some two-sided cell of $\mathbb{A}_n$ such that for every j, $1 < j < r$, either $y_{j-1} \underset{L}{\leq} y_j$ or $y_{j-1} \underset{R}{\leq} y_j$. So by Lemma 14.3, we have either $y_{j-1} \underset{L}{\sim} y_j$ or $y_{j-1} \underset{R}{\sim} y_j$ and then $\sigma(y_{j-1}) = \sigma(y_j)$ for all j, $1 < j < r$. This implies $\sigma(y) = \sigma(w)$. ■

<u>Corollary 14.5</u> <u>The number of two-sided cells of $\mathbb{A}_n$ is equal to the number of partitions of n.</u>

<u>Proof</u> This follows immediately from Theorem 14.4. □

Note that if $x,y \in \mathfrak{P}$ (for $\mathfrak{P}$, see the end of Chapter 13) are in the same two-sided cell of $\mathfrak{P}$ then they are automatically in the same two-sided cell of A_n. So by Theorems 13.2.8 and 14.4, we can describe all the two-sided cells of $\mathfrak{P}$ as well.

Theorem 14.6 For any $\lambda \in \Lambda_n$, the set $\mathfrak{P} \cap \sigma^{-1}(\lambda)$ is a two-sided cell of $\mathfrak{P}$. Any two-sided cell of $\mathfrak{P}$ has such a form. □

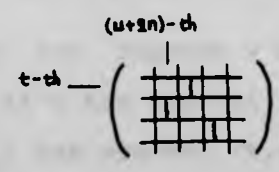
CHAPTER 15 : SOME PROPERTIES OF CELLS AND OTHER EQUIVALENCE CLASSES OF A_n

In this chapter, we shall discuss some properties of the equivalence classes of A_n. Two such properties are particularly interesting: one is the commutativity between a left star operation and a right star operation; the other is the connectness of these equivalence classes.

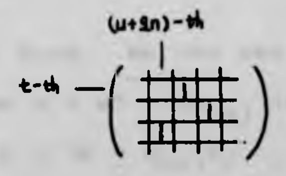
§15.1 THE COMMUTATIVITY BETWEEN A LEFT STAR OPERATION AND A RIGHT STAR OPERATION

Given two integers, t,u, we say $w \in A_n$ has a special (t,u) form if w has one of the following forms.

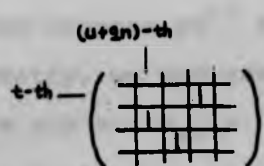
(i)



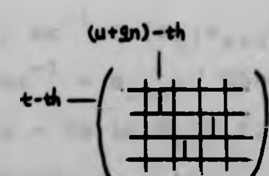
(ii)



(iii)



(iv)



for some $q \in \mathbb{Z}$.

<u>Lemma 15.1.1</u> <u>Assume $w, w' \in A_n, s_t \in \Delta$.</u>

(i) <u>If $w, w' \in \mathcal{D}_R(s_t)$ with $w' = w^*$ in $\mathcal{D}_R(s_t)$, then $\mathcal{L}(w') = \mathcal{L}(w)$.</u>

(ii) <u>If $w, w' \in \mathcal{D}_L(s_t)$ with $w' = {}^*w$ in $\mathcal{D}_L(s_t)$, then $\mathcal{R}(w') = \mathcal{R}(w)$.</u>

<u>Proof</u> (i) $w' = w^*$ in $\mathcal{D}_R(s_t)$ implies $w' \underset{R}{\sim} w$.

By Theorem A, we have $\mathcal{L}(w') = \mathcal{L}(w)$.

(ii) Similarly. □

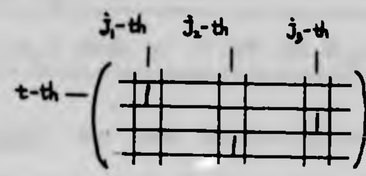
Assume $w \in A_n$, s_t, $s_u \in \Delta$ such that $w \in \mathcal{D}_L(s_t) \cap \mathcal{D}_R(s_u)$. Let $x = w^*$ in $\mathcal{D}_R(s_u)$ and let $y = {}^*w$ in $\mathcal{D}_L(s_t)$. Then by Lemma 15.1.1, we have $x, y \in \mathcal{D}_L(s_t) \cap \mathcal{D}_R(s_u)$. Let $z = {}^*x$ in $\mathcal{D}_L(s_t)$.

<u>Lemma 15.1.2</u> <u>Let $w, x, y, z \in A_n$ be as above. Then $yw^{-1} \neq zx^{-1}$ if and only if w has a special (t,u) form.</u>

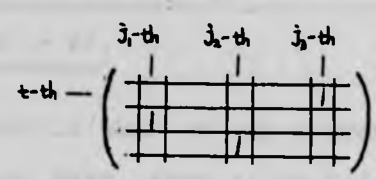
<u>Proof</u> ($\Leftarrow$) Suppose w has a special (t,u) form. We can see that if w has special (t,u) form (i), then $x = w^* = ws_{u+1}$ in $\mathcal{D}_R(s_u)$ has special (t,u) form (iii). So $y = {}^*w = s_{t+1}w$, $z = {}^*x = s_t w$ in $\mathcal{D}_L(s_t)$ have special (t,u) forms (ii), (iv), respectively. Thus $yw^{-1} = s_{t+1} \neq s_t = zx^{-1}$. That is, $yw^{-1} \neq zx^{-1}$. If w has special (t,u) form (ii), (iii) or (iv), then the similar argument can be used.

($\Rightarrow$) Now suppose $yw^{-1} \neq zx^{-1}$. Since $yw^{-1}, zx^{-1} \in \{s_t, s_{t+1}\}$, by symmetry, we may assume $yw^{-1} = s_t$ and $zx^{-1} = s_{t+1}$. That is, $y = s_t w$ and $z = s_{t+1}x$. Since $y = {}^*w$ and $z = {}^*x$ in $\mathcal{D}_L(s_t)$, this implies that w has one of the following forms:

(i')



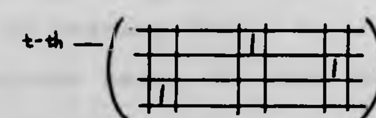
(ii')



and x has one of the following forms:

(i")

(ii")



But $x = w^*$ in $\mathcal{D}_R(s_u)$. So x is obtained from w by transposing two adjacent columns which contain their entries lying in the block A(w), where A(w) consists of three rows of w from the t-th to the (t+2)-th. In case (i'), these two adjacent columns of w must be the j_1-th and the j_2-th columns and then by the fact that $x = w^*$ in $\mathcal{D}_R(s_u)$, w must have special (t,u) form (iv). In case (ii'), these two adjacent columns must be the j_2-th and the j_3-th columns and then by the fact that $x = w^*$ in $\mathcal{D}_R(s_u)$, w must have special (t,u) form (iii). Our proof is complete. □

<u>Proposition 15.1.3</u> <u>Assume $w \in \mathcal{D}_L(s_t) \cap \mathcal{D}_R(s_u)$ for some $s_t, s_u \in \Delta$. Let $x = w^*$ in $\mathcal{D}_R(s_u)$, $x' = {}^*x$ in $\mathcal{D}_L(s_t)$, $y = {}^*w$ in $\mathcal{D}_L(s_t)$ and $y' = y^*$ in $\mathcal{D}_R(s_u)$. Then $x' = y'$.</u>

<u>Proof</u> We see in the proof of Lemma 15.1.2 that if any element $z \in \mathcal{D}_L(s_t) \cap \mathcal{D}_R(s_u)$ has a special (t,u) form, then z^* in $\mathcal{D}_R(s_u)$ and $^*z$ in $\mathcal{D}_L(s_t)$ also have special (t,u) forms.

First assume that w has special (t,u) form (i), then $x = w^* = ws_{u+1}$ in $\mathcal{D}_R(s_u)$ has special (t,u) form (iii) and so $x' = {}^*x = s_t x = s_t w s_{u+1}$ in $\mathcal{D}_L(s_t)$ has special (t,u) form (iv). On the other hand, $y = {}^*w = s_{t+1}w$ in $\mathcal{D}_L(s_t)$ has special (t,u) form (ii) and so $y' = y^* = ys_u = s_{t+1}ws_u$ in $\mathcal{D}_R(s_u)$ has special (t,u) form (iv). Since the process of passing from w to x' or from w to y' only involves the transposes among the rows from the t-th to the (t+2)-th and among the columns from the u-th to the (u+2)-th, we have $x' = y'$. Similarly for the cases when w has special (t,u) form (ii), (iii) or (iv).

Next assume that w has no special (t,u) form. By Lemma 15.1.2, we have $yw^{-1} = x'x^{-1}$ (1). Also since w^{-1} has no special (u,t) form, we see by Lemma 15.1.2 that $x^{-1}(w^{-1})^{-1} = y'^{-1}(y^{-1})^{-1}$. (2) From (1) and (2), we get $y' = yw^{-1}x = x'$.

So in any case, we always have $x' = y'$. Our result is proved. □

Proposition 15.1.3 tells us that left star operations commute with right star operations on any element of A_n. This implies the following result.

<u>Corollary 15.1.4</u> <u>Assume that y, w are in A_n with $y \underset{P_L}{\sim} w$. Then $R(w) = R(y)$. If y, w are also in $D_R(s_t)$ for some $s_t \in \Delta$, let $y' = y^*$, $w' = w^*$ in $D_R(s_t)$. Then $y' \underset{P_L}{\sim} w'$.</u>

<u>Proof</u> By Theorems D and A, we see that $y \underset{P_L}{\sim} w$ implies $y \underset{L}{\sim} w$ and then $R(w) = R(y)$. For the remainder of this corollary, we may assume that y, w are in $D_L(s_u)$ with $y = {}^*w$ in $D_L(s_u)$ for some $s_u \in \Delta$. By Proposition 15.1.3, we see that $y' = ({}^*w)^* = {}^*(w^*) = {}^*w'$ in $D_R(s_t) \cap D_L(s_u)$. So $y' \underset{P_L}{\sim} w'$. □

§15.2 <u>CONNECTNESS OF CELLS AND OTHER EQUIVALENCE CLASSES OF A_n</u>

In this section, we shall discuss the connectness of cells of A_n. The definition of a connected (resp. left connected; right connected) set given here is in a pure algebraic sense rather than in a geometric sense, although it initially comes from geometry when A_n is regarded as the set of alcoves in an affine Euclidean space of dimension n-1. The reason for this is as follows. We want to discuss three kinds of sets: a left connected set, a right connected set and a connected set. But only a left connected set can be described explicitly in a geometric way. Now from the algebraic point of view, we can see that essentially a right connected set is entirely similar to a left connected set, and a connected set is only a natural generalization of the former two kinds of sets.

The connectness is one of the important properties in the structure of cells of A_n.

Definition 15.2.1 We say a set $M \subset A_n$ is left connected if, for any $x,y \in M$, there exists a sequence $x_o = x, x_1, \ldots, x_t = y$ in M such that for every j, $1 < j < t$, $x_j = s_{i_j} x_{j-1}$ with some $s_{i_j} \in \Delta$. Similarly, we can define a right connected set by $x_j = x_{j-1} s_{i_j}$ instead of $x_j = s_{i_j} x_{j-1}$ for each j, $1 < j < t$, as above. We say a set $N \subset A_n$ is connected if for any $x,y \in N$, there exists a sequence $z_o = x, z_1, \ldots, z_u = y$ in N such that for every j, $1 < j < u$, either $x_j = s_{i_j} x_{j-1}$ or $x_j = x_{j-1} s_{i_j}$ with some $s_{i_j} \in \Delta$.

Clearly, any left (resp. right) connected set of A_n is connected.

Our main result in this section is as follows.

Theorem 15.2.2

(i) Any left cell of A_n is left connected and is also a maximal left connected component in the two-sided cell containing it.

(ii) Any right cell of A_n is right connected and is also a maximal right connected component in the two-sided cell containing it.

(iii) Any two-sided cell of A_n is connected.

Remark 15.2.3 For any sets M, N with $M \subset N \subset A_n$, we say M is a maximal left (resp. right) connected component in N if M satisfies the following two conditions:

(1) M is a left (resp. right) connected set of A_n.

(2) If K is a left (resp. right) connected set of A_n satisfying $M \subset K \subset N$, then we have $M = K$.

It is easily seen that if M, N are two left (resp. right) connected sets of Λ_n with $M \cap N \neq \emptyset$ then the set $M \cup N$ is also a left (resp. right) connected sets. By this fact, any set of Λ_n can be expressed uniquely as a disjoint union of some maximal left (resp. right) connected components of it.

To show Theorem 15.2.2, we need some lemmas.

Lemma 15.2.4

(i) Every P_L-equivalence class of Λ_n is left connected.

(ii) Every P_R-equivalence class of Λ_n is right connected.

(iii) Every P-equivalence class of Λ_n is connected.

Proof The P_L-equivalence relation is generated by $w \underset{P_L}{\sim} {}^*w$ in $D_L(s_u)$ with $s_u \in \Delta$. If $w \in D_L(s_u)$, then $^*w$ in $D_L(s_u)$ is either $s_u w$ or $s_{u+1} w$. This implies that every P_L-equivalence class is left connected. The remaining results can be proved similarly. □

Lemma 15.2.5 Assume $\lambda = \{\lambda_1 \geq \ldots \geq \lambda_r\} \in \Lambda_n$. Let $w, w' \in N_\lambda$ be such that $w' = L_{E_u}(w)$ with $1 \leq u \leq \lambda_1$. Then there exists a sequence $x_o = w, x_1, \ldots, x_t = w'$ in $\sigma^{-1}(\lambda)$ such that for every j, $1 \leq j \leq t$, $x_j = s_{i_j} x_{j-1}$ with some $s_{i_j} \in \Delta$.

Proof Assume that $\mu = \{\mu_1 \geq \ldots \geq \mu_m\}$ is the dual partition of λ. Assume $\lambda_{k+1} < u \leq \lambda_k$ for some k, $1 \leq k \leq r$, with the convention that $\lambda_{r+1} = 0$. Assume that w has a standard MDC form $(A_r, \ldots, A_1)$ at $i \in \mathbb{Z}$. Then w' has a standard MDC form $(A_r, \ldots, A_1)$ at i+1 and the u-th layer $\bar{e}((w),E_u)$ of w is movable. So the following properties are obvious:

(i) $j_t^u(w) < j_{t+1}^{u-1}(w)$ for any t, $1 \le t < k$.

(ii) $j_k^u(w) < j_1^{u-1}(w)-n$ and so $j_k^u(w) < j_h^v(w)$ for all h, v with $k < h \le r$ and $1 \le v \le \lambda_h$.

(iii) $j_t^v(w') = j_t^v(w)$ for any t, v with $1 \le t \le r$, $1 \le v \le \lambda_t$ and $v \neq u$.

(iv) $j_t^u(w') = j_{t-1}^u(w)$ for any t, $1 < t \le k$, and $j_1^u(w') = j_k^u(w) + n$.

For any h, $1 \le h \le k$, let $f_{hu}(w)$ be the row of w containing the u-th entry of $A_h(w)$ and let $A_h^o(w)$ (resp. $A_h^1(w)$) be the block consisting of the first u-1 rows (resp. the last λ_h-u rows) of $A_h(w)$.

Now we define an element y of $\tilde{A}_n$ which is obtained from w by permuting the row $f_{hu}(w)$ from the bottom to the top in the block $[A_h^o(w), f_{hu}(w)]$ for all h, $1 \le h < k$ and also permuting the row $f_{ku}(w)$ from the bottom to the top in the block $[A_r,\ldots,A_{k+1},A_k^o,f_{ku}(w)]$. Then by properties (i), (ii), we see that there exists a sequence $w_o = w, w_1,\ldots,w_\alpha = y$ such that for every j, $1 \le j \le \alpha$, $w_j = s_{i_j}w_{j-1}$ with $s_{i_j} \in \Delta$ and $\ell(w_j) = \ell(w_{j-1})-1$. By Lemma 3.4, this implies $\lambda = \sigma(w) = \sigma(w_o) \geqslant \sigma(w_1) \geqslant \ldots \geqslant \sigma(w_\ell) = \sigma(y)$.

y has the DC form

$(f_{ku},A_r,\ldots,A_{k+1},A_k^o,A_k^1,f_{k-1,u},A_{k-1}^o,A_k^1,\ldots,f_{1u},A_1^o,A_1^1)$ at i, where each block of y occurring in this form comes from the corresponding block of w under the above transformation from w to y.

It is clear that for every DC block A in this form of y, we have $j_A^h(y) = j_A^h(w)$, $1 \le h \le |A|$.

By definition of the raising operation from w to w', we see that w' can be obtained from y by permuting the row $f_{h,u}(y)$ from the bottom to the top in the block $[A_{h+1}^1(y), f_{hu}(y)]$ for any h, $1 \le h < k$ and permuting the row $f_{ku}(y)$ from the bottom to the top in the block $[\tilde{A}_1^1(y), f_{ku}(y)]$, where $\tilde{A}_1^1(y)$ is the congruent block of $A_1^1(y)$ between the (i+1-n)-th row and the i-th row of y.

Since $j_t^u(w) > j_{t+1}^{u+1}(w)$ for any t with $1 \le t < k$ and $1 \le t+1 \le \mu_{u+1}$, and since $j_k^u(w) > j_1^{u+1}(w) - n$ when $u < \lambda_1$ we see by properties (iii), (iv) that there exists a sequence $y_o = y, y_1, \ldots, y_\beta = w'$ such that for every j, $1 \le j \le \beta$, $y_j = s_{h_j} y_{j-1}$ with $s_{h_j} \in \Delta$ and $\ell(y_j) = \ell(y_{j-1}) + 1$. By Lemma 3.4, this implies $\sigma(y) = \sigma(y_o) \le \sigma(y_1) \le \ldots \le \sigma(y_\beta) = \sigma(w') = \lambda$.

Now we define a sequence $\xi : x_o = w, x_1, \ldots, x_{\alpha+\beta} = w'$ such that for any j, $0 \le j \le \alpha$, we have $x_j = w_j$ and for any h, $\alpha \le h \le \alpha + \beta$, we have $x_h = y_{h-\alpha}$. Then it is clear that for every j, $1 \le j \le \alpha + \beta$, we have $x_j = s_{\ell_j} x_{j-1}$ with $s_{\ell_j} \in \Delta$. To show the elements in the sequence ξ are in $\sigma^{-1}(\lambda)$, it is enough to show that y is in $\sigma^{-1}(\lambda)$.

Assume $y \in \sigma^{-1}(\lambda')$ for some $\lambda' \in \Lambda_n$. We already know $\lambda' \le \lambda$. For the DC form $(f_{ku}, A_r, \ldots, A_{k+1}, A_k^o, A_k^1, f_{k-1,u}, A_{k-1}^o, A_{k-1}^1, \ldots \ldots, f_{1u}, A_1^o, A_1^1)$ at i of y, let $S = S_1 \cup \ldots \cup S_r$, where

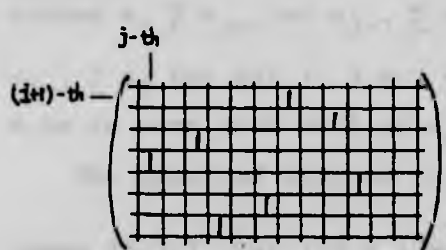
S_h = {all the entries of $A_h(y)$} for $k+1 \le h \le r$.

S_ℓ = {all the entries of $A_{\ell+1}^o(y)$} $\cup$ {the entry of $f_{\ell u}(y)$} $\cup$ {all the entries of $A_\ell^1(y)$} for $1 \le \ell \le k$.

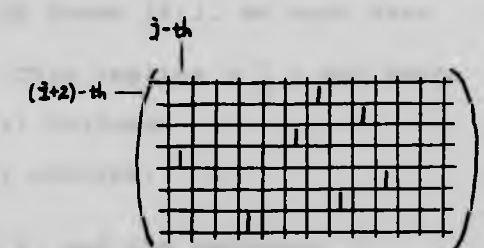
S_k = {all the entries of $\tilde{A}_1^o(y)$} $\cup$ {the entry of $f_{ku}(y)$} $\cup$ {all the entries of $A_k^1(y)$}

where $\tilde{A}_1^o(y)$ is the congruent block of $A_1^o(y)$ between the $(i+1-n)$-th row and the i-th row of y. Then we see that $S = S_1 \cup \ldots \cup S_r$ satisfies $C_n(y,r)$ with $|S_h| = \lambda_h$ for any h, $1 < h < r$. This implies $\lambda' > \lambda$. So $\lambda' = \lambda$ and ξ is the required sequence. □

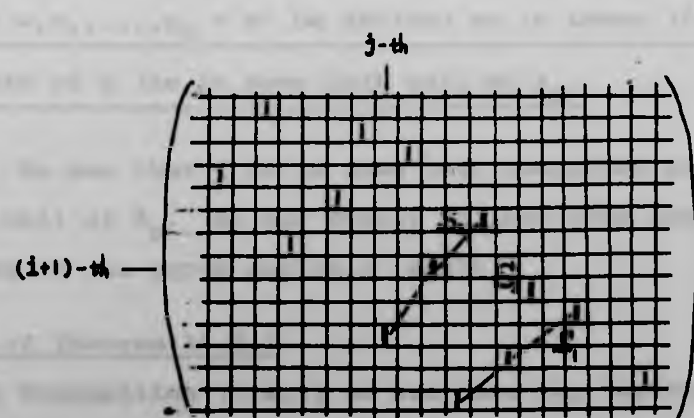
An example for the elements w, w', y in Lemma 15.2.5 is as follows.



w



w'



y

where $n = 7$, $\lambda = \{3 > 3 > 1\}$, $k = 2$ and $u = 2$. $S = S_1 \cup S_2 \cup S_3$ is as defined in the proof of Lemma 15.2.5, where for any h, $1 < h < 3$, S_h consists of all entries of y occurring as the vertices of the corresponding broken line.

<u>Lemma 15.2.6</u> <u>Let K be any two-sided cell of $\tilde{A}_n$.</u>

(i) <u>Any left connected set of K is in some left cell of $\tilde{A}_n$.</u>

(ii) <u>Any right connected set of K is in some right cell of $\tilde{A}_n$.</u>

<u>Proof</u> Assume that $M \subseteq K$ is a left connected set. Let $x, y \in M$. Then there exists a sequence $x_0 = x, x_1, \ldots, x_t = y$ in M such that for every j, $1 < j < t$, $x_j = s_{i_j} x_{j-1}$ with some $s_{i_j} \in \Delta$ and so either $x_j \underset{L}{\leq} x_{j-1}$ or $x_{j-1} \underset{L}{\leq} x_j$. By Lemma 14.3, we must have $x_{j-1} \underset{L}{\sim} x_j$ for all j, $1 < j < t$. This implies $x \underset{L}{\sim} y$ and hence M is in some left cell of $\tilde{A}_n$. (i) follows.

The proof of (ii) is entirely similar. □

<u>Lemma 15.2.7</u> <u>Let $\lambda \in \Lambda_n$, $w, w' \in N_\lambda$ and the sequence</u> <u>$\xi : x_0 = w, x_1, \ldots, x_t = w'$ be defined as in Lemma 15.2.5. Then the elements of ξ lie in some left cell of $\tilde{A}_n$.</u>

<u>Proof</u> We see that ξ is in some left connected set of some two-sided cell of $\tilde{A}_n$. So our result follows from Lemma 15.2.6. □

Now we can prove our main result.

<u>Proof of Theorem 15.2.2</u>

By Proposition 11.4.4, we see that any two elements x, y in the same left cell can be transformed from one to another by a succession of raising operations in N_λ and left star operations, where $\lambda = \sigma(x)$. Thus it follows from Lemmas 15.2.4

of the corresponding broken line.

Lemma 15.2.6 Let K be any two-sided cell of A_n.

(i) Any left connected set of K is in some left cell of A_n.

(ii) Any right connected set of K is in some right cell of A_n.

Proof Assume that $M \subseteq K$ is a left connected set. Let $x,y \in M$. Then there exists a sequence $x_0 = x, x_1, \ldots, x_t = y$ in M such that for every j, $1 < j < t$, $x_j = s_{i_j} x_{j-1}$ with some $s_{i_j} \in \Delta$ and so either $x_j \underset{L}{\leq} x_{j-1}$ or $x_{j-1} \underset{L}{\leq} x_j$. By Lemma 14.3, we must have $x_{j-1} \underset{L}{\sim} x_j$ for all j, $1 < j < t$. This implies $x \underset{L}{\sim} y$ and hence M is in some left cell of A_n. (i) follows.

The proof of (ii) is entirely similar. □

Lemma 15.2.7 Let $\lambda \in \Lambda_n$, $w,w' \in N_\lambda$ and the sequence $\xi : x_0 = w, x_1, \ldots, x_t = w'$ be defined as in Lemma 15.2.5. Then the elements of ξ lie in some left cell of A_n.

Proof We see that ξ is in some left connected set of some two-sided cell of A_n. So our result follows from Lemma 15.2.6. □

Now we can prove our main result.

Proof of Theorem 15.2.2

By Proposition 11.4.4, we see that any two elements x, y in the same left cell can be transformed from one to another by a succession of raising operations in N_λ and left star operations, where $\lambda = \sigma(x)$. Thus it follows from Lemmas 15.2.4

and 15.2.7 that any left cell of A_n is left connected. Then the latter part of (i) follows by Lemma 15.2.6.

The proof of (ii) is entirely similar.

We know that any two-sided cell of A_n is a RL-equivalence class. For any x,y with $x \underset{\Gamma}{\sim} y$, there exists a sequence $x_o = x, x_1, \ldots, x_t = y$ such that for every j, $1 < j < t$, either $x_j \underset{L}{\sim} x_{j-1}$ or $x_j \underset{R}{\sim} x_{j-1}$. So by (i), (ii), we get (iii) easily. □

REFERENCES

[1] D. Kazhdan and G. Lusztig., Representations of Coxeter groups and Hecke algebras. Invent. Math. 53 (1979), 165-184.

[2] G. Lusztig., Some examples of square integrable representations of semisimple p-adic groups. Trans. Amer. Math. Soc. 277 (1983), 623-653.

[3] G. Lusztig., Some problems in the representation theory of finite Chevalley groups. Proc. Symp. Pure Math. (A.M.S) 37 (1980), 313-317.

[4] D. Kazhdan and G. Lusztig., Schubert varieties and Poincaré duality. Proc. Symp. Pure Math. (A.M.S) 36 (1980), 185-203.

[5] G. Lusztig., Characters of reductive groups over a finite field.

[6] G. Lusztig., Cells in affine Weyl groups.

[7] C. Greene., Some partitions associated with a partially ordered set. J. of Comb. Th., vol. 20 1 (1976), 69-79.

[8] D. Vogan., A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra. Math. Ann. 242 (1979), 209-224.

[9] D.N. Verma., Möbius inversion for the Bruhat ordering on a Weyl group. Ann. Scient. Éc. Norm. Sup. (1971), 393-399.